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Discrete Mathematics: Chapter 6, Functions & Equivalence Relations

Calvin Jongsma

Dordt College, calvin.jongsma@dordt.edu

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Discrete Mathematics: Chapter 6, Functions & Equivalence Relations

Abstract

Modern science and contemporary Western culture are unthinkable without high-level mathematics. Quantitative modes of thinking, mathematical ideas, algorithmic techniques, and symbolic reasoning permeate the way we conceptualize and interact with the world today. After number and its use in computation, the notion of function, usually expressed in terms of a symbolic formula, is probably the most pervasive mathematical idea used in scientific applications. Functions help formulate important causal connections between different types of measures in numerous scientific fields, and they provide a way to compare different algebraic structures in advanced mathematics.

Our interest in functions here will be different than it is in calculus. Calculus graphs functions and explores certain features of those graphs such as local extreme values or the area under a curve by making certain specialized function calculations. Here we will instead investigate some general algebraic features of functions that come up in various discrete mathematics applications as well as in more advanced areas of mathematics.

Keywords

functions, algebra, algorithms, set theory, binary system

Disciplines

Christianity | Computer Sciences | Mathematics

Comments

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- © 2016 Calvin Jongma

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✠ Chapter 6 ✠

FUNCTIONS
&
EQUIVALENCE RELATIONS

6.1 Functions and Their Properties

Modern science and contemporary Western culture are unthinkable without high-level mathematics. Quantitative modes of thinking, mathematical ideas, algorithmic techniques, and symbolic reasoning permeate the way we conceptualize and interact with the world today. After number and its use in computation, the notion of function, usually expressed in terms of a symbolic formula, is probably the most pervasive mathematical idea used in scientific applications. Functions help formulate important causal connections between different types of measures in numerous scientific fields, and they provide a way to compare different algebraic structures in advanced mathematics.

Our interest in functions here will be different than it is in calculus. Calculus graphs functions and explores certain features of those graphs such as local extreme values or the area under a curve by making certain specialized function calculations. Here we will instead investigate some general algebraic features of functions that come up in various discrete mathematics applications as well as in more advanced areas of mathematics. Some discrete mathematics texts also probe algorithmic properties of functions, such as the complexity of calculations needed to compute function values; we will not pursue this topic due to time constraints.

After spending two sections on functions, we will introduce the slightly broader notion of relation. This will connect up with some ideas mentioned earlier in the text and will provide a theoretical foundation for the material in the following chapter.

Brief Historical Background

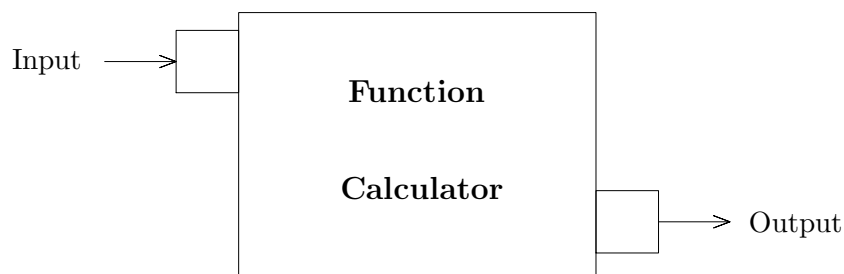
The concept of a function first emerged in a prominent way during the Scientific Revolution when calculus was being developed in the last quarter of the seventeenth century. Leibniz coined the term *function*, but many people besides him worked with the idea. Even prior to that time, scientists worked with the ideas of direct and inverse proportionality, ideas whose roots go back into ancient times. The function concept, however, broadened this idea. One quantity was taken to be a function of other quantities if it depended upon them in some definite manner that could be calculated from them. Functional relations between quantities could thus be expressed via a formula. Given an equation relating variable quantities, algebra, geometry, and especially calculus could be employed to tease out further connections among them.

The concept of a function was originally associated with expressions calculated via ordinary numerical operations, such as addition, multiplication, exponentiation, or even taking roots. The most common functions could be formulated using a finite number of operations, but infinite sums and products were also considered legitimate operations, even though criteria for convergence were initially lacking. This expanded the realm of possible functions beyond polynomials and rational functions; trigonometric functions, for example, could be expressed using infinite series.

Mathematicians also associated functions with graphs. A function was eventually considered continuous if its graph was a connected curve. Every function had its graph; did every continuous graph also represent a function? Would a single formula encapsulate the relationship between the variables? Disputes over just what the realm of functions included and what they could be used to analyze broke out in the middle of the eighteenth century as mathematicians used solutions of differential equations to treat wave phenomena in physics. Before the issue was fully clarified, even stranger functions arose in connection with Fourier's work on heat. By the middle of the nineteenth century, however, a fairly general notion of function had been accepted, one that allowed mathematicians to settle the various earlier disputes and make further progress in real and complex analysis. This is the notion of function we will adopt in our work below.

Toward a Definition of Function

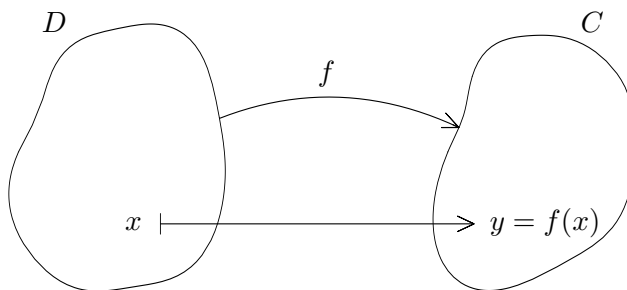
There are several different ways we can think about functions. Perhaps the easiest way in our computer age is one that employs the ideas and terminology of *input* and *output*. A function can be considered a machine or a mechanical procedure or a rule that, given an appropriate input, produces a uniquely determined output. A function thus provides a deterministic linkage between two sets of values, inputs and outputs. If x stands for the input and y for the output, we often write $y = f(x)$ to indicate that the output value y is produced by the function f from the input x .



While mathematicians and textbooks occasionally get lazy and talk about $f(x)$ as if it were the function, a more precise way of speaking distinguishes the function f from its output $f(x)$. Given a formula that expresses a functional relationship between x and y , such as $y = x^2$, we don't often think about the function as a mathematical object since it seems so ephemeral. We have the input, and we get the output; but what is the function? It is the procedure that produces the output from the input. More advanced mathematics and computer science courses, however, treat functions as more than this, as mathematical objects in their own right. In fact, mathematicians also abstract from the computational process itself. How outputs are calculated is largely irrelevant from an abstract mathematical perspective (though that might be the main interest for a computer scientist): what's important about a function is how the outputs correspond to their inputs, how the function assigns output values to the inputs. We thus have the following informal definition of a function. Blob diagrams like the one below the definition are often used to picture such a functional correspondence.

DEFINITION 6.1 - 1: Function (Informal Version)

- f is a **function** from a set D to a set C iff f is a correspondence assigning a unique value $f(x)$ in C to each value x in D ; in symbols, $f : D \rightarrow C$ iff $(\forall x \in D)(\exists! y \in C)(y = f(x))$.
- The set D of all inputs is called the **domain** of the function. The set C of all potential outputs is called the **codomain** of the function; the set of all actual outputs is called the **range** of the function.
- Functions are also called **maps**, **mappings**, or **transformations**. If $y = f(x)$, y is called the **value of f at x** or the **image of x under the mapping f** , and x is called the **pre-image of y** . The functional assignment is sometimes indicated by the notation $f : x \mapsto y$.



Mathematically speaking, what is important about a function in general is that it is a relationship or association in which every input value corresponds to some definite output value. Specific functions may have other extra features, but nothing more is needed in order for a relationship to qualify as a function. Thus, mathematicians tend to think of a function in abstract terms as a directed correspondence between sets of elements. There may or may not be any causal or dynamic relationship between the variables, and the relationship may or may not be given by a single formula or an equation.

✠ **EXAMPLE 6.1 - 1**

Explain why the following define functions.

- a) The formula for converting degree measure into radian measure is given by $r = \frac{\pi}{180} d$.
- b) Let $P(x)$ denote the refund/income tax payment calculated on a tax form for a given year that is owed to/by the person whose social security number is x .

Solution

- a) The conversion formula $r = \frac{\pi}{180} d$ stipulates r as a function of d . The opposite conversion formula $d = \frac{180}{\pi} r$ specifies the same relationship between r and d , only this time from the other direction, so it defines a different function. In both cases, the domain and codomain can be taken to be the set \mathbb{R} of real numbers.
- b) Since a definite amount of money is either owed to or by each person with a social security number, P is a function from the set of social security numbers to the set of positive and negative numbers given to two decimal places. P is a function even though there is no set formula for calculating $P(x)$; in fact, while a complicated equation might be devised, it would be of no use whatsoever as a formula. It only summarizes a discrete set of data; it would not cover any new cases that arise. On the other hand, if x indicated taxable income instead of social security numbers, a usable formula could be found, though it would not be a single equation. Currently, the tax percentage changes as income increases; there are different levels of taxation (formulas) depending on income level.

✠ **EXAMPLE 6.1 - 2**

Determine whether the following formulas or descriptions define functions from some set D to a set C . Unless specified, assume that all variables involved range over real numbers.

- a) $x - 3y = 7$.
- b) $s = 16t^2$.
- c) $x^2 + y^2 = 1$.
- d) $\chi_S(x) = \begin{cases} 0 & : x \notin S \\ 1 & : x \in S \end{cases}$ for any $S \subseteq \mathbb{R}$.
- e) $M(S)$ is the minimum value of S , $S \subseteq \mathbb{N}$.

Solution

- a) The formula $x - 3y = 7$ does define a function from \mathbb{R} to \mathbb{R} . Given any real number x we can find a unique real number y associated with it. The formula $y = (x - 7)/3$ gives the value of y explicitly and shows that each y -value is unique, whatever real number its pre-image x is.
- b) The formula $s = 16t^2$ also defines a function from \mathbb{R} to \mathbb{R} . Given any real value t , a unique real number s is associated with it via this transformation.
- c) $x^2 + y^2 = 1$ may or may not define a functional relation between x and y . There are several complications here. First of all, it does not define a function from \mathbb{R} to \mathbb{R} because not all real values for x are permitted. If $|x| > 1$, $x^2 + y^2 \neq 1$, regardless of the choice taken for y . We can fix

this problem by restricting our inputs to the closed interval $[-1, 1]$. Now it is possible for real y -values to be found satisfying the equation.

However, even with this restriction, $x^2 + y^2 = 1$ does not define a function from $[-1, 1]$ to \mathbb{R} . The problem is that some (almost all) x are associated with more than one y . For example, the points $(0, 1)$ and $(0, -1)$ both satisfy the equation. So there is not a unique image y for a given pre-image x . We can remedy this by ruling out all negative y -values. The formula $x^2 + y^2 = 1$ now defines y as function of x from the closed interval $[-1, 1]$ into the non-negative real numbers $\mathbb{R}^+ \cup \{0\}$.

As this example makes clear, whether something is a function or not very much depends on what the domain and codomain are taken to be. These two sets are essential parts in the definition of a function.

- d) $\chi_S: \mathbb{R} \rightarrow \mathbb{R}$ defines the mapping known as the *characteristic function of S* . This function assigns the value 0 (no) to any number outside S and the value 1 (yes) to each number inside S . It is defined piecewise, but that doesn't make it any less a bona-fide function. It satisfies all the requirements: all real numbers are assigned a unique value.

When $S = \mathbb{Q}$, we have a well known function due to Dirichlet, the nineteenth-century mathematician who first proposed thinking of functions as correspondences. Dirichlet's function is often used as an example in an advanced analysis course in connection with generalizing the theory of integration beyond what is learned in introductory calculus.

- e) This description gives a function from $\mathcal{P}(\mathbb{N})$ into \mathbb{N} . To each subset of \mathbb{N} (each element of the power set of \mathbb{N}) there is associated its least element, which exists, is unique, and is a natural number. Functions do not need to be restricted to ones with number inputs, though that is the case in most lower-level mathematics courses. They can be defined for inputs of any kind (cf. Exercise 2). What is crucial is that each element of the domain corresponds to a unique element of the codomain.

You know from earlier mathematics courses how to graph functions from D to C : the first value, from the domain D , is plotted along the horizontal axis; the second value, from the codomain C , is plotted along the vertical axis. Applying the definition of a function to its graph, we obtain the following **vertical line criterion** for identifying when a curve is the graph of a function: *A graph is the graph of a function iff for each input x , a vertical line drawn at x passes through exactly one point of the graph.* This holds because there must be exactly one y -value associated with any given x -value.

This criterion (and the other graphic criteria we give below) gives us an informal way to determine whether a curve is the graph of a function, but *it cannot function as the sole basis of a rigorous argument.* This is because graphs are never made by plotting all possible outputs but are drawn by extrapolating from a finite amount of information.

The notion of a graph does suggest a set-theoretically precise way to think about functions, however. Since the abstract mathematical definition of a function only focuses on which outputs are related to which inputs, the action of a function is completely determined by which ordered pairs are graphed. This gives the following more formal definition of a function. We will not use this definition much in our work (it takes a while to get used to thinking about functions as sets of ordered pairs), but it illustrates a tendency we identified earlier, namely, to treat everything in terms of Set Theory.*

DEFINITION 6.1 - 2: Function (Formal Set-Theoretic Version)

A function F from a set D to a set C is a set of ordered pairs inside $D \times C$ such that every x in D is the first element of some unique pair (x, y) belonging to F ; in symbols, $F: D \rightarrow C$ iff $F \subseteq D \times C \wedge (\forall x \in D)(\exists! y \in C)((x, y) \in F)$.

* A completely opposite approach to functions arises in Category Theory, a more recent subfield of Abstract Algebra. There functions are taken as undefined, and they are used to help define sets.

Definition of Function, Equality of Functions

Equality of functions can be taken in two slightly different ways, depending on how a function is viewed. One way derives from a minimalist view of functions. If a function is simply the assignment of outputs to inputs (or a set of ordered pairs), then functions are equal iff the same outputs correspond to the same inputs. According to this approach, the domains and ranges of equal functions must be identical, but their codomains need not be, since these don't enter explicitly into the correspondence enacted by the function.

This is not completely satisfactory. When we discuss composition of functions in Section 6.2, we only need the codomain of the first function to be the same as the domain of the second. The range of the function is less important here than the codomain. Moreover, we may not immediately know just what the range of a particular function is, though we may be able to specify a codomain quite easily. Finally, unless we distinguish range and codomain, all functions will automatically be onto functions (see below).

We thus take the position that the codomain of a function is an important piece of information and so ought to be included in the way we conceptualize functions. Our approach will therefore be that of the following definition.

DEFINITION 6.1 - 3: Function (Final Version)

A **function** consists of a domain D and a codomain C together with an assignment f from D into C such that each element x of D corresponds to a unique y in C .

This expanded outlook is the one favored in algebraic circles. According to this definition, two functions are identical iff three things hold true: the functions have (1) the same domain D , (2) the same codomain C , and (3) the same correspondence action/set of ordered pairs.

PROPOSITION 6.1 - 1: Equality of Functions

Two functions $f_1: D_1 \rightarrow C_1$ and $f_2: D_2 \rightarrow C_2$ are equal, denoted by $f_1 = f_2$, iff $D_1 = D_2$, $C_1 = C_2$, and $(\forall x \in D_1)(f_1(x) = f_2(x))$.

Proof:

This result is an immediate consequence of our definition. ■

✂ EXAMPLE 6.1 - 3

Show, as indicated, that the following real-valued functions of a real variable are not equal.

- $2^x \neq x^n$ for any n
- $\lfloor x + 1 \rfloor \neq \lceil x \rceil$

Solution

Here the domains and codomains agree, so we must show that the functions' actions differ.

- Suppose $2^x = x^n$ for some n .

Then these must agree for all inputs x .

They obviously disagree already for $x = 0$ and $x = 1$: $1 \neq 0$ and $2 \neq 1$.

But they also disagree if we exclude these inputs from our domain.

Taking $x = 2$, we obtain $2^2 = 2^n$, which forces $n = 2$.

However, $2^x \neq x^2$: for $x = 1$, we'd need $2 = 1$.

Thus $2^x \neq x^n$; exponential functions are not power functions.

- $\lfloor \]$ is the *floor function*, returning the greatest integer less than or equal to its input, while $\lceil \]$ is the *ceiling function*, yielding the least integer greater than or equal to its input.

Letting $x = 0$, we get $1 = \lfloor 0 + 1 \rfloor \neq \lceil 0 \rceil = 0$. In fact, these two functions disagree on all integer values, though they agree everywhere else.

You will rarely need to show that two concrete functions are equal; showing they are unequal is the more normal situation, as in the last example, and there it is usually because the actions differ. However, in order to develop a mathematical theory involving functions, we will often need to show functions are equal. We will see examples of this in Section 6.2 (see also Exercises 38 and 39 on decomposing real-valued functions of a real variable).

One-To-One and Onto Functions

Functions have two very important set-theoretic properties: being *one-to-one*, and being *onto*. We will define and illustrate these properties here; their significance will become more apparent when we discuss composite and inverse functions in Section 6.2. In addition to these properties, we will define the algebraic property of *monotonicity* and show how it is related to the other properties.

DEFINITION 6.1 - 4: One-to-One Functions

A function f is a **one-to-one** or **injective** function iff different inputs yield different outputs; i.e., iff $\forall x_1 \forall x_2 (x_1 \neq x_2 \rightarrow f(x_1) \neq f(x_2))$; or equivalently, via *Contraposition*,
 iff $\forall x_1 \forall x_2 (f(x_1) = f(x_2) \rightarrow x_1 = x_2)$.

Thus, according to this definition, a function is one-to-one iff every element y of the codomain has *at most one* pre-image x in the domain. It need not be related to any x -value, of course (remember, the codomain is not the range), but if there is one, it must be unique.

On the other hand, every x -value must have a unique y -value for any function whatsoever, whether or not the function is one-to-one: this is part of the definition of a function. *Be careful*, therefore, *not to confuse* the uniqueness requirement for being a function (having *unique images*/being a many-to-one relation) with the *additional uniqueness requirement* for being one-to-one (having *unique pre-images*).

If a function is given by an equation, you can sometimes determine whether it is one-to-one by trying to solve the given equation for x in terms of y and seeing how many solutions are possible. As we will see in Section 6.2, this strategy is also valuable for finding a formula for a function's inverse, if it has one.

The definition for being one-to-one can be translated into a **horizontal line criterion**: *A function is one-to-one iff a horizontal line through any y -value of its codomain meets the graph in at most one point.* This is true because there is either no x (if y is not an output) or else there is exactly one x associated with that y . However, this criterion *does not provide us with a rigorous method* for demonstrating that a function is one-to-one.

✠ EXAMPLE 6.1 - 4

Determine whether the following functions are one-to-one (injective).

- $f: \mathbb{R} \rightarrow \mathbb{R}$, defined by $y = f(x)$, where $x - 3y = 7$.
- $s: \mathbb{R} \rightarrow \mathbb{R}$, defined by $s(t) = 16t^2$.
- $M: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{N}$, defined by $M(S)$ is the minimum value of S , for $S \subseteq \mathbb{N}$.
- $\lfloor \cdot \rfloor: \mathbb{R} \rightarrow \mathbb{Z}$, defined by $\lfloor x \rfloor =$ the greatest integer less than or equal to x .

Solution

- The function f is a one-to-one function.

We will show this by applying the definition.

First note that $y = f(x) = \frac{x - 7}{3}$.

Suppose that $f(x_1) = f(x_2)$.

Then $\frac{x_1 - 7}{3} = \frac{x_2 - 7}{3}$

Multiplying this equation by 3 and then adding 7 gives $x_1 = x_2$.

Hence f is one-to-one.

This can also be argued in the following way, by solving for x .

Let y be any element in the range of the function.

Then there is at most one x from the domain associated with it:

$$x = 7 + 3y \text{ defines } x \text{ uniquely in terms of } y.$$

So f must be one-to-one.

b) The position function s is not one-to-one.

To show this, it suffices to find a counterexample: two x -values that yield the same y .

Since squaring is involved, this is easy: $f(-1) = 16 = f(1)$.

In fact, s is nearly a two-to-one function instead of a one-to-one function; only 0 has a unique pre-image.

However, if the domain of s were modified to consist only of non-negative real numbers, then s would be one-to-one. The pre-image of a given real number y would either be non-existent (for $y < 0$) or it would be given by $t = \frac{\sqrt{y}}{4}$.

c) The function M is not one-to-one.

Let $S_1 = \{0\}$ and let $S_2 = \{0, 1\}$. The minimum element of both of these sets is 0, so different pre-images have the same image.

In fact, each image (minimum value) has infinitely many pre-images, so the function is infinitely-many-to-one.

d) The greatest integer function is also not one-to-one. For example, all the real numbers in the interval $[0, 1)$ return 0 as their output.

DEFINITION 6.1 - 5: Onto Functions

A function $f: D \rightarrow C$ is an **onto** or **surjective** function iff every element of C has a pre-image; i.e., iff $(\forall y \in C)(\exists x \in D)(f(x) = y)$.

Thus, according to this definition, f is *onto* iff the range of f is the same as its codomain C . Every element y of the codomain is related to *at least one* element x in the domain. An element of C might be related to more than one input, but it must have at least one pre-image.

If a function f is given by an equation, the *Method of Analysis* can often be used to show that f is onto: assume that an arbitrary y from the codomain is related to some x in the domain by means of the equation, and then solve the equation to get x in terms of y . You should then check that your solution x is an element of the domain and that it yields y .

There is also a **horizontal line criterion** for being an onto function: *A function is onto iff a horizontal line through any y -value of its codomain meets the graph of f in at least one point.* This holds because every y must be an output in order for the function to be onto. Again, this criterion does not provide a rigorous method for testing whether a function is onto.

✠ **EXAMPLE 6.1 - 5**

Determine whether the following functions are onto (surjective).

a) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $y = f(x)$, where $x - 3y = 7$.

b) $s: \mathbb{R} \rightarrow \mathbb{R}$ defined by $s(t) = 16t^2$.

c) $M: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{N}$ defined by $M(S) = \min(S)$, the minimum value of S , for $S \subseteq \mathbb{N}$.

d) $\lfloor \cdot \rfloor: \mathbb{R} \rightarrow \mathbb{Z}$, defined by $\lfloor x \rfloor =$ the greatest integer less than or equal to x .

Solution

a) The function f is an onto function; $f(x) = y = \frac{x - 7}{3}$.

Let y be any real number. We must find an x -value associated with y .

Since $x = 7 + 3y$ gives x in terms of y and since such an x is real when y is real, x lies in the domain of the function.

Checking whether this x is the pre-image of y , we see that $f(7 + 3y) = \frac{(7 + 3y) - 7}{3} = y$.

So f is onto.

- b) The function s is not onto.
 To show this, it suffices to find a y -value without an associated x -value.
 Since the codomain is \mathbb{R} , this is easy: take any negative number, such as -1 . This will have no pre-image in \mathbb{R} .
 Note, however, that if we restricted the codomain of s to the non-negative real numbers, then s would be onto. Whether a function is onto, then, just like whether a function is one-to-one, depends upon both the domain and codomain.
- c) The function M is surjective.
 Given any $n \in \mathbb{N}$, we must find a set so that n is its minimum.
 But this is easy: let $S = \{n\}$.
- d) The floor function is onto. Given any integer y , $\lfloor y \rfloor = y$.

DEFINITION 6.1 - 6: One-to-One-and-Onto Functions

A function $f: D \rightarrow C$ is a **one-to-one-and-onto** or **bijective** function iff f is a one-to-one function from D onto C /is an injective and surjective function.

We have already met up with one-to-one-and-onto functions. We called them *one-to-one correspondences* in the context of our work with numerosity for sets. Here are some reformulations of what we did there, using our new terminology.

$S \sim T$ iff there is a bijective function $f: S \rightarrow T$.

$S \preceq T$ iff there is an injective function $f: S \rightarrow T$.

$S \prec T$ iff there is an injective function $f: S \rightarrow T$ but no bijective function from S to T .

If a function f is one-to-one-and-onto, then given any y in the codomain of f , we must be able to find *exactly one* x in the domain of f that corresponds to it. If f is presented by means of an equation, then we may be able to show that it is one-to-one-and-onto by solving the equation for x in terms of y and arguing that each such x is unique.

The graphic criteria given above can be combined to give a **horizontal line criterion** for functions that are one-to-one-and-onto: *A function is one-to-one-and-onto iff a horizontal line passing through any y -value of the codomain meets the function's graph in exactly one point.* This is so because every y -value is an output (image) and is associated with a unique input (pre-image) x -value. This criterion helps us identify one-to-one-and-onto functions informally, but a rigorous argument requires using the definition.

✠ **EXAMPLE 6.1 - 6**

Determine which of the following functions are one-to-one-and-onto.

- a) $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $y = f(x)$, where $x - 3y = 7$.
 b) $s: \mathbb{R} \rightarrow \mathbb{R}$ defined by $s(t) = 16t^2$.
 c) $M: \mathcal{P}(\mathbb{N}) \rightarrow \mathbb{N}$ defined by $M(S) = \min(S)$, the minimum value of S , $S \subseteq \mathbb{N}$.
 d) $\lfloor \cdot \rfloor: \mathbb{R} \rightarrow \mathbb{Z}$, defined by $\lfloor x \rfloor =$ the greatest integer less than or equal to x .

Solution

- a) The function f is both one-to-one and onto. See Examples 4 and 5.
 b) Our work above shows that s is neither one-to-one nor onto.
 If the domain were restricted to non-negative reals and the codomain to non-negative reals, s would then be both one-to-one and onto.
 c) The function M is onto but not one-to-one.
 d) The greatest integer function $\lfloor \cdot \rfloor$ is also onto but not one-to-one.

✠ **EXAMPLE 6.1 - 7**

Show that $y = f(x) = \frac{x}{x+3}$ is one-to-one onto its range, and determine the range.

Solution

No domain is given; we will assume it is the largest set of real numbers for which the formula makes sense, namely, $\mathbb{R} - \{-3\}$.

To prove that f is one-to-one, we will show that the pre-images of two equal images are themselves equal.

So suppose $\frac{x_1}{x_1 + 3} = \frac{x_2}{x_2 + 3}$.

Cross-multiplying and simplifying yields $x_1 = x_2$.

Thus f is one-to-one.

f is obviously onto its range by definition.

To determine the range, we must see what y -values result from real x -values different from 3.

Suppose $y = \frac{x}{x + 3}$.

All such y are real numbers, but not all real numbers can be put into this form. To see which ones can, we will solve this equation for x . This will show explicitly which x -values in the domain, if any, can generate a given y -value.

Cross-multiplying, $xy + 3y = x$, so $x(1 - y) = 3y$.

Hence $x = \frac{3y}{1 - y}$, which is defined for all y except 1. So $y = 1$ has no pre-image x .

However, given any real number $y \neq 1$, these x -values will produce it:

$$\frac{\frac{3y}{1 - y}}{\frac{3y}{1 - y} + 3} = \frac{3y}{3y + (3 - 3y)} = \frac{3y}{3} = y.$$

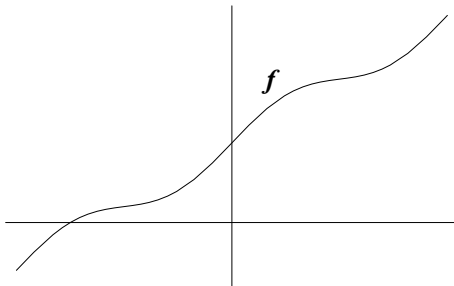
Thus, the only y -value that must be excluded from \mathbb{R} is 1. This would give $1 = \frac{x}{x + 3}$,

leading to $3 = 0$, which is false.

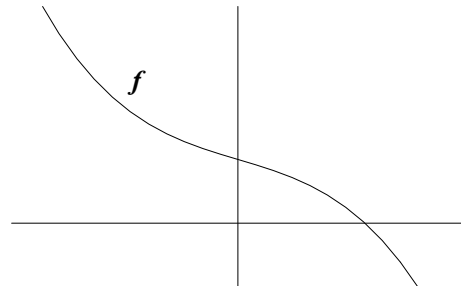
Therefore the range of f is $\mathbb{R} - \{1\}$.

Note that here $y = 1$ is the equation of the asymptote for the function f .

An important class of real-valued functions of a real variable are monotone functions. These are those functions whose images are ordered in the codomain the same way as their pre-images are in the domain (monotone increasing functions) or they are in the reverse order (monotone decreasing functions).



Monotone Increasing Function



Monotone Decreasing Function

DEFINITION 6.1-7: Monotone Functions

Let D be a domain of real numbers.

- A function $f: D \rightarrow \mathbb{R}$ is monotone increasing iff for every $x_1 < x_2$ in D , $f(x_1) < f(x_2)$.
- A function $f: D \rightarrow \mathbb{R}$ is monotone decreasing iff for every $x_1 < x_2$ in D , $f(x_1) > f(x_2)$.
- A function $f: D \rightarrow \mathbb{R}$ is monotone iff it is either monotone increasing on D or monotone decreasing on D .

One-to-one functions need not be monotone (see Exercise 28), but monotonic functions have to be one-to-one, as the next proposition asserts.

PROPOSITION 6.1 - 2: Monotonicity and One-to-One Functions

If $f: D \rightarrow \mathbb{R}$ is a monotone function defined on a real-valued domain D , then f is a one-to-one function.

Proof:

This can be proved by taking cases and using the first characterization of a one-to-one function. See Exercise 27. ■

Images, Pre-Images, and Set Theory (Optional)

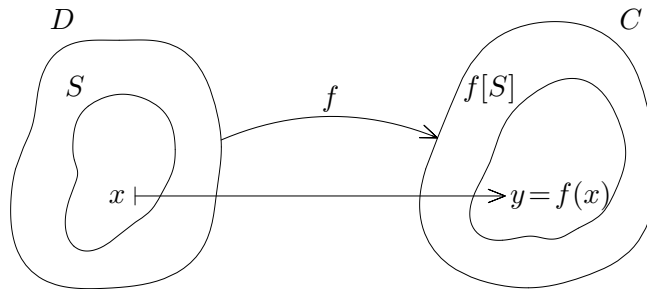
The following results show how functions in general interact with various set-theoretic operations and with the natural order-relation \subseteq of sets. In a couple of cases, if the functions are further specified as being either one-to-one or onto, sharper conclusions can be concluded (see Exercises 30b and 33).

We will not have occasion to use these results in the rest of the text, but they get rather heavy use in advanced analysis, topology, and abstract algebra. Working with some of them here will give you some practice in abstract reasoning with functions. To help you develop your arguments, draw some abstract function diagrams (blob diagrams, as below) so you can visualize where the object or set is that you are reasoning about.

DEFINITION 6.1 - 8: Image of a Set

Let $f: D \rightarrow C$ be a function from D into C with $S \subseteq D$. Then the **image of S under the function f** , denoted by $f[S]$, is the set of images of elements of S :

$$f[S] = \{y \in C : y = f(x) \text{ for some } x \in S\}.$$

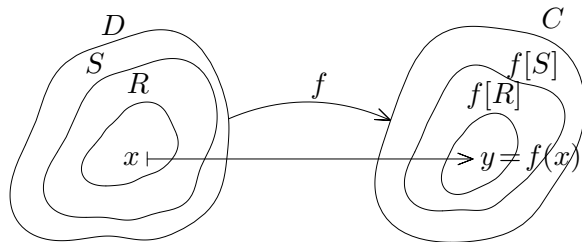


PROPOSITION 6.1 - 3: Images of Subsets

If $f: D \rightarrow C$ is any function with $R \subseteq S \subseteq D$, then $f[R] \subseteq f[S]$.

Proof:

Suppose $R \subseteq S \subseteq D$, and let y be any element of $f[R]$. Then $y = f(x)$ for some $x \in R$. But since $R \subseteq S$, $x \in S$, too. Hence $y = f(x)$ is in $f[S]$. Therefore $f[R] \subseteq f[S]$. ■



PROPOSITION 6.1 - 4: Images of Unions, Intersections, and Set Differences

Let $f : D \rightarrow C$ be any function with $R \subseteq D$ and $S \subseteq D$. Then the following hold:

- a) $f[R \cup S] = f[R] \cup f[S]$
- b) $f[R \cap S] \subseteq f[R] \cap f[S]$
- c) $f[R - S] \supseteq f[R] - f[S]$

Proof:

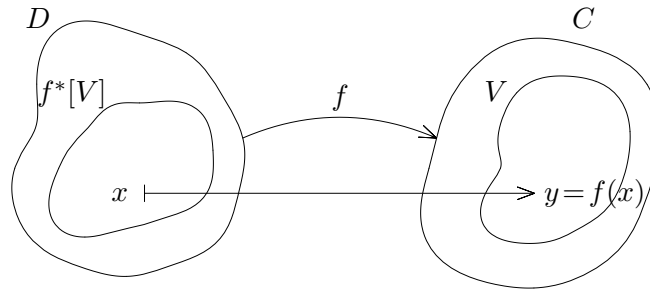
- a) See Exercise 29a.
- b) See Exercise 29b.
- c) See Exercise 29c. ■

The above inclusions for intersection and set difference are as good as can be gotten in general; see Exercise 30ac.

DEFINITION 6.1 - 9: Pre-Image of a Set

Let $f : D \rightarrow C$ be a function from D into C with $V \subseteq C$. Then the **pre-image of V under the function f** , denoted by $f^*[V]$, is the set of pre-images of elements of V :

$$f^*[V] = \{x \in D : f(x) \in V\}.$$



PROPOSITION 6.1 - 5: Pre-Images of Subsets

If $f : D \rightarrow C$ is any function with $U \subseteq V \subseteq C$, then $f^*[U] \subseteq f^*[V]$.

Proof:

See Exercise 32. ■

PROPOSITION 6.1 - 6: Pre-Images of Unions, Intersections, and Set Differences

Let $f : D \rightarrow C$ be any function with $U \subseteq C$ and $V \subseteq C$. Then the following hold:

- a) $f^*[U \cup V] = f^*[U] \cup f^*[V]$
- b) $f^*[U \cap V] = f^*[U] \cap f^*[V]$
- c) $f^*[U - V] = f^*[U] - f^*[V]$

Proof:

- a) See Exercise 34a.
- b) See Exercise 34b.
- c) See Exercise 34c. ■

Note that while we cannot conclude equality for certain image sets (see *Proposition 4bc*), the same is not true for pre-images. No additional conditions need to be added to the function to guarantee the equality of the sets involved.

In addition to seeing how the image and pre-image operators interact with the operations of *Set Theory*, we can also explore how they interact. Two general results can be gotten, which can each be sharpened if the function involved is a special type (see Exercise 37).

PROPOSITION 6.1 - 7: Pre-Images of Images; Images of Pre-Images

Let $f : D \rightarrow C$ be any function with $S \subseteq D$ and $V \subseteq C$. Then the following hold:

- a) $f^*[f[S]] \supseteq S$
- b) $f[f^*[V]] \subseteq V$

Proof:

- a) See Exercise 36a.
- b) See Exercise 36b. ■

EXERCISE SET 6.1

Problems 1-6: Some Simple Functions

Determine whether the following are functions and whether they have the properties of being one-to-one or onto. Justify your answers by using the relevant definitions.

1. Let $\mathcal{S}(x) = x + 1$ be the successor function on \mathbb{N} .
 - a. Explain why \mathcal{S} is a function from \mathbb{N} into \mathbb{N} .
 - b. Is \mathcal{S} a one-to-one function? Is \mathcal{S} an onto function? Is \mathcal{S} a one-to-one-and-onto function?
2. Let S be the set of all possible compound SL sentences having P and Q as their atomic constituent sentences, and let B be the set $\{T, F\}$. Define $v : S \rightarrow B$ by $v(X) = T$ if X is true, and $v(X) = F$ if X is false.
 - a. If $v(P) = T$ and $v(Q) = F$, what is $v(\neg P)$? $v(P \wedge Q)$? $v(P \vee Q)$? $v(P \rightarrow Q)$? $v(P \leftrightarrow Q)$?
 - b. Explain why v is a function. How does v relate to what we did in our study of SL?
 - c. Is v a one-to-one function? Is v an onto function? Is v one-to-one-and-onto?
- *3. Let $B = \{0, 1\}$ and $B^2 = B \times B$. Define $f : B^2 \rightarrow B$ by $f(x, y) = x \cdot y$. Answer the following questions, with reasons.
 - a. Does f define a function from $B^2 \rightarrow B$?
 - b. Is f one-to-one? Is f onto? Is f one-to-one-and-onto?
- *4. Let $f(x) = 2x + 4$.
 - a. Explain why f is a function from \mathbb{R} to \mathbb{R} .
 - b. Is f a one-to-one function? Is f an onto function? Is f a one-to-one-and-onto function? Explain.
- *5. Let $A(x) = |x|$.
 - a. Explain why A is a function from \mathbb{R} to \mathbb{R} . Give a piecewise-defined formula for $A(x)$ by considering the cases when $x \geq 0$ and $x < 0$.
 - b. Is A an injection? Is A a surjection? Is A a bijection? Explain.
6. Let S be any set and let $D = \{(x, x) : x \in S\}$. Answer the following questions, with reasons.
 - a. Does D define a function from S into S ? What is its formula? Why is D called the identity function on S ?
 - b. Is D a one-to-one function? Is D onto? Is D one-to-one and onto?

Problems 7-12: True or False

Are the following statements true or false? Explain your answer.

- *7. $\lfloor x + y \rfloor = \lfloor x \rfloor + \lfloor y \rfloor$.
8. $\lfloor mx \rfloor = m \cdot \lfloor x \rfloor$ for $m \in \mathbb{Z}$.
9. $\lfloor xy \rfloor = \lfloor x \rfloor \cdot \lfloor y \rfloor$.
10. Two functions are equal if their domains, ranges, and actions all agree.
- *11. If f is any function, then $x_1 = x_2$ only if $f(x_1) = f(x_2)$.
12. A function is one-to-one iff each input has a unique output.

Problems 13-14: Explanations

Define the following terms using your own words.

- *13. One-to-one function
- 14. Onto function

Problems 15-23: More Complex Functions

Determine whether the following are functions and whether they have the properties of being one-to-one or onto. Justify your answers by using the relevant definitions.

- *15. Let $p(x) = x^2 - 3x + 2$.
 - a. Explain why this equation defines a function $p: \mathbb{R} \rightarrow \mathbb{R}$.
 - b. Is p a one-to-one function? Is it onto? Is it a bijection?
- 16. Let $p(x) = x^2 - 3x + 2$, as in Problem 15.
 - a. Does this equation define a function $p: \mathbb{R} \rightarrow [-1/4, +\infty)$?
 - b. Is f a one-to-one function? Is it onto? Is it a bijection? Hint: to explore whether it is onto, use the method of completing the square.
- 17. Let $p(x) = x^2 - 3x + 2$, as in Problem 15.
 - a. Does this equation define a function $p: [3/2, +\infty) \rightarrow [-1/4, +\infty)$?
 - b. Is p one-to-one? Is it onto? Is it a bijection? Hint: to explore whether it is one-to-one, use the method of completing the square to solve for x and explain why the value you get is unique.
 - c. What do Problems 15–17 exhibit about one-to-one-and-onto functions?
- 18. Let $c(x) = 2x^3 - 1$.
 - a. Explain why c defines a function from \mathbb{R} to \mathbb{R} .
 - b. Is c one-to-one? Is it onto? Is it a bijection?
- 19. Let $g(x) = \frac{1 + e^x}{1 - e^x}$.
 - a. Explain why g is a function from \mathbb{R}^* into \mathbb{R} .
 - b. Is g one-to-one? Is it onto? Is it a bijection?
- *20. Find, with reasons, functions $f: \mathbb{N} \rightarrow \mathbb{N}$ that satisfy the following conditions:
 - a. A function that is neither one-to-one nor onto.
 - *b. A function that is one-to-one but not onto.
 - *c. A function that is onto but not one-to-one.
 - d. A function that is both one-to-one and onto.
- *21. *Functions on Finite Sets*
Suppose that $f: S \rightarrow S$ is a function from a finite set S to itself.
 - *a. Can f be one-to-one without being onto? If so, give an example. If not, explain why it can't.
 - b. Can f be onto without being one-to-one? If so, give an example. If not, explain why it can't.
 - c. Answer parts *a* and *b* if the domain and codomain are different finite sets instead of the same set.
- 22. *Coordinate Projections*
The projection function $P_1: \mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by $P_1(x, y, z) = x$.
 - a. Explain why P_1 is a function from 3-space into 1-space.
 - b. Is P_1 one-to-one? Is it onto? Is it one-to-one-and-onto?
- 23. *Residues modulo n*
Let \mathbb{Z} be the set of all integers and $\mathbb{Z}_n = \{0, 1, \dots, n - 1\}$. Define $r: \mathbb{Z} \rightarrow \mathbb{Z}_n$ by $r(m) =$ the remainder when m is divided by n .
 - a. Explain why r is a function from \mathbb{Z} to \mathbb{Z}_n .
 - b. Is r a one-to-one function? Is it onto? Is it a bijection?

Problems 24-26: Counting Numbers of Functions

Count the number of functions of the types indicated in the following problems.

- *24. Consider the set F of all possible functions from $S = \{1, 2\}$ to $S = \{1, 2\}$.
- List all such functions in ordered pair format and give them names f_1, f_2 , etc. How many functions are there from S into S ?
 - Classify each of the functions identified in part *a* as being one-to-one, onto, both one-to-one and onto, or neither.
 - Which of your functions in part *a* are monotone increasing? monotone decreasing?
- *25. Consider the set F of all possible functions from $S = \{1, 2, \dots, n\}$ to $T = \{1, 2, \dots, m\}$.
- How many functions are there from S into T ? Explain. Why?
 - The set of all functions from S into T is often denoted by T^S . Explain why this is a good notation.
- EC c. How many of the functions in part *a* are one-to-one? onto? one-to-one-and-onto? Give reasons for your answers. Take different cases where necessary.
Hint: counting formulas from Chapter 4 will help here.

26. Functions on Infinite Sets

- Prove that if D is countably infinite and $f: D \rightarrow C$ is an injection, then its range $f[D]$ is countably infinite.
- Is the conclusion of part *a* true for surjective functions? for functions in general?
- If f is an injection, how is the cardinality of the range $f[D]$ related to that of D for a set D of any size whatsoever? Explain.

Problems 27-28: Monotonicity and Algebraic Properties of Functions

The following problems explore relations holding between monotonicity and being one-to-one or onto.

- *27. *Proposition 2*
Prove *Proposition 2*: If $f: D \rightarrow \mathbb{R}$ is a monotone function defined on a real-valued domain D , then f is a one-to-one function. Argue the case of a monotone increasing function in detail, and then explain how that can be used or modified for the case of a monotone decreasing function.
- EC 28. *Converse to Proposition 2*
Show that the converse to *Proposition 2* is false: one-to-one functions need not be monotonic.

Problems 29-31: Images and Algebraic Properties of Functions

The following problems explore relations holding between set images and being one-to-one or onto.

- *29. *Proposition 4*
Let $f: D \rightarrow C$ be any function with $R \subseteq D$ and $S \subseteq D$.
- Prove *Proposition 4a*: $f[R \cup S] = f[R] \cup f[S]$.
 - Prove *Proposition 4b*: $f[R \cap S] \subseteq f[R] \cap f[S]$.
 - Prove *Proposition 4c*: $f[R - S] \supseteq f[R] - f[S]$.
30. *Strengthened Versions of Proposition 4*
- Give a counterexample to show that set equality cannot be asserted in the conclusion of *Proposition 4b*: find sets R and S so that $f[R] \cap f[S] \not\subseteq f[R \cap S]$.
 - Can equality be asserted in the conclusion of *Proposition 4b*, yielding $f[R \cap S] = f[R] \cap f[S]$, if f is either one-to-one or onto? If so, prove it; if not, give a counterexample.
 - Give a counterexample to show that set equality cannot be asserted in the conclusion of *Proposition 4c*: find sets R and S so that $f[R - S] \not\subseteq f[R] - f[S]$.
31. *Images of Disjoint Sets*
Prove or disprove the following results. If a result is false, give a counterexample and explain why it is false. If requiring the function to be one-to-one or onto will convert a false statement into a true one, modify the result accordingly and then prove it.
- Suppose $f: D \rightarrow C$, $R \subseteq D$, and $S \subseteq D$. Then $R \cap S = \emptyset$ only if $f[R] \cap f[S] = \emptyset$.
 - Suppose $f: D \rightarrow C$, $R \subseteq D$, and $S \subseteq D$. Then $R \cap S = \emptyset$ if $f[R] \cap f[S] = \emptyset$.

Problems 32-35: Pre-Images and Algebraic Properties of Functions

The following problems explore relations holding between set pre-images and being one-to-one or onto.

- *32. *Proposition 5*
Prove *Proposition 5*: If $f: D \rightarrow C$ is any function with $U \subseteq V \subseteq C$, then $f^*[U] \subseteq f^*[V]$.
33. *Sufficient Conditions for a Strengthened Proposition 5*
Can equality be asserted in the conclusion of *Proposition 5*, yielding $f^*[U] = f^*[V]$, if f is either one-to-one or onto? If so, prove it; if not, give a counterexample.
34. *Proposition 6*
Let $f: D \rightarrow C$ be any function with $U \subseteq C$ and $V \subseteq C$.
a. Prove *Proposition 6a*: $f^*[U \cup V] = f^*[U] \cup f^*[V]$.
b. Prove *Proposition 6b*: $f^*[U \cap V] = f^*[U] \cap f^*[V]$.
c. Prove *Proposition 6c*: $f^*[U - V] = f^*[U] - f^*[V]$.
35. *Pre-images of Disjoint Sets*
Prove or disprove the following results. If a result is false, give a counterexample and explain why it is false. If requiring the function to be one-to-one or onto will convert a false statement into a true one, modify the result accordingly and then prove it.
a. Suppose $f: D \rightarrow C$, $U \subseteq C$, and $V \subseteq C$. Then $U \cap V = \emptyset$ only if $f^*[U] \cap f^*[V] = \emptyset$.
b. Suppose $f: D \rightarrow C$, $U \subseteq C$, and $V \subseteq C$. Then $U \cap V = \emptyset$ if $f^*[U] \cap f^*[V] = \emptyset$.

Problems 36-37: Images and Pre-Images of One Another

The following problems explore relations among set images and set pre-images.

36. *Proposition 7*
Let $f: D \rightarrow C$ be any function with $S \subseteq D$ and $V \subseteq C$. Then the following hold:
a. Prove *Proposition 7a*: $f^*[f[S]] \supseteq S$.
b. Prove *Proposition 7b*: $f[f^*[V]] \subseteq V$.
37. *Strengthened Versions of Proposition 7*
EC a. What additional condition on f allows equality to be asserted in *Proposition 7a*, yielding $f^*[f[S]] = S$? Prove your claim.
b. What additional condition on f allows equality to be asserted in *Proposition 7b*, yielding $f[f^*[V]] = V$? Prove your claim.

Problems 38-39: Decomposing Functions

The following problems look at ways to decompose functions into other related functions whose compound function equals the original one.

- EC 38. *Positive and Negative Parts of a Function*
Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be any real-valued function of a real variable and let $f_-: \mathbb{R} \rightarrow \mathbb{R}$ and $f_+: \mathbb{R} \rightarrow \mathbb{R}$ be defined as follows:

$$f_+(x) = \begin{cases} f(x) & : f(x) \geq 0 \\ 0 & : f(x) < 0 \end{cases} \qquad f_-(x) = \begin{cases} f(x) & : f(x) \leq 0 \\ 0 & : f(x) > 0 \end{cases}$$

Prove that $f_s: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f_s(x) = f_-(x) + f_+(x)$ is equal to f .

39. *Even and Odd Functions*

Definition: A real-valued function f of a real variable is **even** iff $f(-x) = f(x)$ for all x in the domain of f . A real-valued function f of a real variable is **odd** iff $f(-x) = -f(x)$ for all x in the domain of f .

- a. Prove that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial function having all even powers, then f is an even function.
b. Prove that if $f: \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial function having all odd powers, then f is an odd function.
c. If $f: \mathbb{R} \rightarrow \mathbb{R}$ is a polynomial function having both even and odd powers, is f even, odd, both, or neither? Explain.
EC d. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be any real-valued function of a real variable. Determine two functions f_e and f_o such that f_e is even, f_o is odd, and $f_e(x) + f_o(x) = f(x)$.
Hint: first suppose such functions exist (*Method of Analysis*). Then determine what they must be.

HINTS TO STARRED EXERCISES 6.1

3. a. Review the definition of *function*.
b. Review the definitions for *one-to-one* functions and *onto* functions.
4. a. Look at the domain and codomain of f .
b. Review the definitions for *one-to-one* functions and *onto* functions.
5. a. In developing the formula for $A(x)$ when x is negative, be careful not to use absolute value notation: that's what you're defining. Find another way to get the associated positive magnitude out of a negative input.
b. To determine whether A is a surjection, remember that the codomain is \mathbb{R} , not the set of non-negative reals.
7. [No hint.]
11. [No hint.]
13. [No hint.]
15. a. Review the definition of *function*.
b. To determine whether p is *onto*, take note of the codomain of p .
20. b. Many linear and quadratic functions work for this one.
c. Modify the function from Exercise 5 to fit your needs here.
21. a. Keep in mind that S is finite.
24. a. Remember that two functions f_1 and f_2 are equal when, for all x in the domain, $f_1(x) = f_2(x)$.
b. Review the definitions of *one-to-one* functions and *onto* functions.
c. Review the definitions of *monotone increasing* and *monotone decreasing* functions.
25. a. Note that in designing a function, each element of S can potentially be paired with m elements from T .
b. Review your answer from part a .
27. To show that f is *one-to-one*, use the negative form of the definition: suppose that $x_1 \neq x_2$ and then show that $f(x_1) \neq f(x_2)$.
29. a. Remember the strategies used to prove two sets are equal.
32. Remember the strategy used to prove one set is a subset of another.

6.2 Composite and Inverse Functions

Given two functions, it may be possible to combine them in different ways to create more complicated functions. If the domains and codomains of the two functions agree and if the codomain supports arithmetic, we may define arithmetic operations on the functions by point-wise operations on their images. For example, if $f: \mathbb{R} \rightarrow \mathbb{R}$ is given by $f(x) = x^2$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = \sqrt[3]{x}$, then the function $f + g: \mathbb{R} \rightarrow \mathbb{R}$ can be defined by the equation $(f + g)(x) = x^2 + \sqrt[3]{x}$ and the function $f \cdot g$ by $(f \cdot g)(x) = x^2 \cdot \sqrt[3]{x}$.

Each of the following compound functions is well-defined by the following definition since the arithmetic operations on the outputs are themselves well-defined, yielding unique real values.

DEFINITION 6.2-1: Arithmetic Operations on Functions

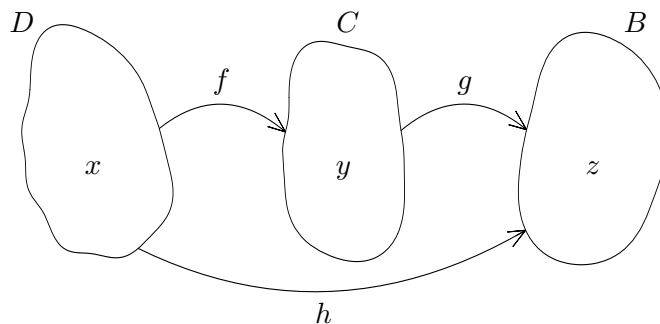
Let $f: D \rightarrow \mathbb{R}$ and $g: D \rightarrow \mathbb{R}$ be two real-valued functions of a real variable sharing a common domain D and let c be a real number. Then $f + g$, $f - g$, $c \cdot f$, $f \cdot g$, and f/g denote the following functions from D to \mathbb{R} :

- a) $(f + g)(x) = f(x) + g(x)$
- b) $(f - g)(x) = f(x) - g(x)$
- c) $(c \cdot f)(x) = c \cdot f(x)$
- d) $(f \cdot g)(x) = f(x) \cdot g(x)$
- e) $(f/g)(x) = f(x)/g(x)$, provided $g(x) \neq 0$.

We can also decompose compound functions into simpler functions. Given a function such as $h(x) = 3x - 1$, we can decompose h into the difference $f - g$ of the two functions $f(x) = 3x$ and $g(x) = 1$. The function f is the product of the constant function $c(x) = 3$ and the identity function $i(x) = x$, so we can decompose h even further: $h = c \cdot i - g$. Such decompositions are important for a variety of purposes, including determining computational complexity of functions. We will not pursue any of this here. We will instead focus on a more algebraic operation, composition of functions, along with the related ideas of identities and inverses. Decomposing functions will play a minor role when we compute function inverses.

Composition of Functions Defined

In addition to arithmetic operations on functions, there is another operation that is more set-theoretic or algebraic in nature. Given two functions that hook up properly, we can connect them in series, as it were, to create a composite function. Combining them in this way requires that the outputs of the first be available as inputs for the second. This is satisfied if the codomain of the first function agrees with the domain of the second.



So, if $f: D \rightarrow C$ and $g: C \rightarrow B$ are functions in which the codomain of f equals the domain of g , then the assignment $h(x) = g(f(x))$ defines a function $h: D \rightarrow B$. For, given any $x \in D$, there is a unique $y \in C$ such that $y = f(x)$, since f is a function. Similarly, since g is a function, $g(f(x))$ is a unique output in B . Thus each input x from D yields a unique output $z = g(f(x))$ in B , guaranteeing that h is a function from D into B . This legitimizes the following definition.

DEFINITION 6.2-2: Composite Functions

If $f: D \rightarrow C$ and $g: C \rightarrow B$, then the **composite function** f followed by g is the function $g \circ f: D \rightarrow B$ whose action is given by $(g \circ f)(x) = g(f(x))$.

Note that the action of the composite function $g \circ f$ consists of f 's action followed by that of g . The reverse-looking order in which this is written is due to the fact that we write the input to the right of the function symbol. To indicate that f acts first on an x , we put it closest to x , and we put g to the left of f to show that it acts on $f(x)$. So while we write first g and then f (reading from left to right), the order of composition goes in the opposite direction.

Composition is the most fundamental operation on functions there is. It can be performed even when it makes no sense to talk about doing arithmetic on the functions' images or pre-images.

Nevertheless, composition cannot always be performed. If two functions f and g link up properly, with the domain of g being the codomain of f , then composition yields a new function. But if they do not so link up, you cannot compose them. Since composition is not universally defined, it is only a *partial operation* on functions.

However, we can still investigate those matters that pertain to composable functions: what algebraic properties does composition have? How is composition related to the one-to-one and onto properties of functions studied in Section 6.1? Is there an identity element for the operation of composition? If so, are there also inverses? Do all functions have inverses, or just some? These are the sorts of things we will explore in the rest of Section 6.2.

Basic Algebraic Properties of Composition

We will begin with a negative proposition of sorts: the order in which functions are performed makes a difference. This is intuitively clear from other contexts: the composite action of putting on your underwear and your outer-wear must obviously be done in a certain order to achieve a desired affect. The same is true of functions. Composition is not generally commutative, not even if both composite functions are defined and have the same domains and codomains. This result is sometimes formulated by writing $g \circ f \neq f \circ g$, but this equation can be somewhat misleading since we do not mean to assert that the two composite functions *must be* different (see, for example Exercise 3), only that they are not always the same. A more careful formulation includes the quantifiers and puts negation in the proper place, as is done in the next proposition.

PROPOSITION 6.2-1: Composition is not Commutative

$$\neg \forall f \forall g (g \circ f = f \circ g).$$

Proof:

The following simple counterexample establishes our claim (nearly any choice will work: look for one of your own before continuing).

The functions $f(x) = x + 1$, $g(x) = 2x$ from \mathbb{N} into \mathbb{N} do not commute:

$(g \circ f)(x) = 2x + 2$ and $(f \circ g)(x) = 2x + 1$ have the same domain and codomain, but their actions differ: for instance, $(g \circ f)(1) \neq (f \circ g)(1)$ because $4 \neq 3$. ■

While the commutative property fails to hold for composition, the associative property does hold. A sequence of three operations can be done by combining the operations in two different ways, but they both yield the same result.

PROPOSITION 6.2-2: Composition is Associative

Suppose f , g , and h are functions that can be composed in the order given. Then

$$(h \circ g) \circ f = h \circ (g \circ f).$$

Proof:

That the domains and codomains of the two composite functions agree is obvious: the domain is that of f , the codomain that of h .

The actions are also the same:

$$\begin{aligned} ((h \circ g) \circ f)(x) &= (h \circ g)(f(x)) = h(g(f(x))), \quad \text{and also} \\ (h \circ (g \circ f))(x) &= h((g \circ f)(x)) = h(g(f(x))). \end{aligned}$$

Thus the two functions are equal. ■

Composition of Special Types of Functions

In Section 6.1 we discussed the properties of functions being one-to-one and onto. We will now see how composition relates to these properties. This will help us investigate inverse functions below.

PROPOSITION 6.2-3: Composition of One-to-One and Onto Functions

Let $f: D \rightarrow C$ and $g: C \rightarrow B$, so that $g \circ f: D \rightarrow B$.

- If f and g are both **one-to-one** functions (injections), then so is $g \circ f$.
- If f and g are both **onto** functions (surjections), then so is $g \circ f$.
- If f and g are both **one-to-one-and-onto** functions (bijections), then so is $g \circ f$.

Proof:

- To show that $g \circ f$ is one-to-one, we will work our way back from B to D through C . You may want draw a blob diagram to help you visualize and follow the proof. Suppose, then, that $(g \circ f)(x_1) = (g \circ f)(x_2)$ in B . Then $g(f(x_1)) = g(f(x_2))$ by the definition of composition. Since g is a one-to-one function, $f(x_1) = f(x_2)$ in C . Since f is also a one-to-one function, $x_1 = x_2$ in D . Thus $g \circ f$ is one-to-one, too.
- This part proceeds similarly. See Exercise 12.
- This is an immediate consequence of the last two parts. ■

The converses of *Proposition 3* are all false, but partial converses do hold (see Exercise 13).

Identity Functions and Their Properties

An identity element for a binary operation is one that leaves all elements unaffected when it operates upon them. For example, 0 is the additive identity for the integers, and 1 is the integers' multiplicative identity. We will now explore the notion of an identity for composition of functions.

For composition to be a genuine binary operation, it must be defined for all pairs of functions in our universe of discourse. Thus, it is necessary to restrict the collection of functions in our universe to ones whose domain and codomain are some common set S . This condition is also sufficient, provided the universe is *closed* under composition; i.e., provided that the composition of any two functions in the collection is also in the collection.

If we fix a set S and consider the collection of all functions $f: S \rightarrow S$, then composition is a binary operation. Here it makes sense to ask whether an identity element exists and what properties it might have. We first define an identity before showing that one exists.

DEFINITION 6.2-3: Identity Element for Composition

A function i is an **identity element** for the operation of composition defined on a collection \mathcal{F} of functions iff $i \circ f = f = f \circ i$ for all f in \mathcal{F} .

If such a function exists, it is fairly obvious that it must be unique. This is proved in the usual way (cf. Example 2.4-10): you assume that i_1 and i_2 both denote identities and then prove that they are equal (see Exercise 16).

Observe from *Definition 3* that if an identity i exists for composition, then $i(f(x)) = (i \circ f)(x) = f(x)$, so i must leave all images $f(x)$ unchanged. That is, i acts as the “do-nothing” function on these elements. We will show that such a function is indeed the identity element for composition, after first defining what *identity functions* are.

DEFINITION 6.2-4: Identity Function

The **identity function** on a set S is the function $I_S: S \rightarrow S$ defined by $I_S(x) = x$ for all x .

The identity function I_S is obviously a one-to-one-and-onto function (see Exercise 17). It is also the unique identity element for composition defined on the class of all functions from S into S , as the next proposition claims.

PROPOSITION 6.2-4: Identity Functions are Identity Elements for Composition

The identity function I_S is the identity element for composition in the collection of all functions $f: S \rightarrow S$.

Proof:

Use function equality to show that I_S satisfies *Definition 3*. See Exercise 18. ■

Inverse Functions and Their Properties

For sets that contain identities we can ask whether inverses also exist. Two objects are inverses relative to a given operation iff together they yield the identity: 3 and -3 are additive inverses in the set of integers because they add up to 0, the additive identity; 3 has no multiplicative inverse in the set of integers, but $1/3$ is its multiplicative inverse in the set of rational numbers because $3 \cdot 1/3 = 1$, the multiplicative identity.

We can explore this same concept of inverses for functions under composition. It turns out that some functions have inverses and some don't. We'll see below that whether or not a function has an inverse depends upon the properties of a function's being one-to-one and onto.

DEFINITION 6.2-5: Inverse Functions

If $f: D \rightarrow C$ and $g: C \rightarrow D$, then f and g are **inverse functions** of one another relative to composition iff $g \circ f = I_D$ and $f \circ g = I_C$.

The final output after composing inverses is thus the same as the original input; inverse functions undo one another. The following proposition is merely a mild point-wise rephrasing of the definition.

PROPOSITION 6.2-5: Inverse Functions Undo One Another

If $f: D \rightarrow C$ and $g: C \rightarrow D$, then f and g are inverses of each other iff $g(f(x)) = x$ for all x in D and $f(g(y)) = y$ for all y in C .

Proof:

Use *Definition 4* and *Definition 5*. See Exercise 26. ■

✧ **EXAMPLE 6.2-1**

Show that the function $g: \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(y) = \frac{y-7}{3}$ is an inverse for the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x + 7$.

Solution

We will use *Proposition 5* to show this.

$$g(f(x)) = g(3x + 7) = \frac{(3x + 7) - 7}{3} = x, \text{ and}$$
$$f(g(y)) = f\left(\frac{y-7}{3}\right) = 3 \cdot \left(\frac{y-7}{3}\right) + 7 = y.$$

Thus g is f 's inverse. ■

At this point, we haven't determined when inverses exist, but when they do, they are unique. Arguing this uses a typical uniqueness proof strategy.

PROPOSITION 6.2-6: Uniqueness of Inverses

If $f: D \rightarrow C$ has an inverse, then it is unique.

Proof:

Suppose that f has two inverses $g_1: C \rightarrow D$ and $g_2: C \rightarrow D$.

To show $g_1 = g_2$, we only need to show that they have the same action on any input y ; their domains (C) and codomains (D) clearly agree.

Let y be any element of C .

By *Proposition 5*, since g_1 is an inverse for f , $y = f(g_1(y))$.

Since g_2 is an inverse for f , applying g_2 to y yields $g_2(y) = g_2(f(g_1(y))) = g_1(y)$.

Thus $g_1 = g_2$. ■

Since there is only one inverse for a function f when it has one, we can speak of *the* inverse of f and give it a standard symbol: f^{-1} denotes f 's unique inverse function.

Caution: $f^{-1} \neq 1/f$ (see Exercise 28a). f^{-1} denotes the *inverse for f relative to the operation of composition of functions*, while $1/f$ denotes its *multiplicative inverse* (which makes sense only if multiplication and division are defined for C and $f(x) \neq 0$). We could invent another notation for the composition inverse, but exponents are handy to use for repeated compositions and their inverses just as they are for repeated multiplication and their inverses (see Exercises 19–24).

We will now prove the *Fundamental Theorem on Inverse Functions*. We will formulate and prove this characterization without using exponent notation for inverse functions so it is clear that we are *not illegally assuming* g to be f 's inverse prior to proving it in the second part of the proof. This result is a strengthened biconditional version of *Proposition 5*.

THEOREM 6.2-1: Fundamental Theorem on Inverse Functions

If $f: D \rightarrow C$ and $g: C \rightarrow D$, then f and g are inverses iff $\forall x \forall y (y = f(x) \leftrightarrow x = g(y))$.

Proof:

Before we prove this theorem, let's analyze what it says. Suppose that f and g is each a function whose domain matches the other's codomain. Then f and g are inverses, according to this theorem, iff a universal "undoing" biconditional is satisfied: $y = f(x) \leftrightarrow x = g(y)$. The logical form of this proposition is complicated enough to warrant consciously adopting a game plan. Analyzing the sentential structure of the proposition, we see that it is roughly in the form $P \wedge Q \rightarrow (R \leftrightarrow (S \leftrightarrow T))$. While this is a rather complex logical form, a *Backward-Forward Proof Analysis* readily generates an overall proof strategy to implement.

We will begin by supposing $P \wedge Q$ for the purpose of CP . Then we will use BI : first supposing R , we will prove $S \leftrightarrow T$, which itself needs BI . Next, supposing $S \leftrightarrow T$ and using BE , we will prove R . This will finish off the consequent of our conditional; the theorem will follow via CP . To indicate the different level of subproofs in our argument, we will indent them.

Suppose, then, that f and g are two functions, as given.

Now suppose in the first place that f and g are inverses.

- Suppose moreover that $y = f(x)$.

[We will show that $g(y) = x$ by applying g to $y = f(x)$.]

Since g is a function, $g(y) = g(f(x))$; but $g(f(x)) = x$ since g is f 's inverse.

Thus, $g(y) = x$.

- Suppose now that $x = g(y)$.

A proof that $y = f(x)$ proceeds in precisely the same way: apply f to both sides of the given equation and use the fact that f is a function and is g 's inverse.

On the other hand, suppose that $y = f(x) \leftrightarrow x = g(y)$ holds for functions f and g .

[We show that f and g are inverses by showing that they undo each other.]

- Let $y = f(x)$ be any image of f .

Applying the function g , we get $g(y) = g(f(x))$.

But $g(y) = x$ (applying BE to the condition given), so $g(f(x)) = x$.

- Similarly, if $x = g(y)$, it follows that $f(g(y)) = y$.

Therefore, by *Proposition 5*, f and g are inverses. ■

Calculating Inverse Functions

We will now develop methods for computing inverse functions. We showed that two functions were inverses in Example 1, but they were already given to us. What can we do to find them in the first place?

There are two main approaches to calculating formulas for inverse functions, both based on the *Fundamental Theorem on Inverse Functions*.

Two functions f and g are inverses iff they undo one another's actions: $f: x \mapsto y$ iff $g: y \mapsto x$. Thus, to determine the behavior of the inverse function, you need to undo the operation of the given function. Now, if you can decompose the action of $y = f(x)$ into a sequence of consecutive simple steps, each of which can be reversed, then the formula $x = g(y)$ can be computed by a **Simple Reversal Method**: *calculate an inverse function by performing the inverse operations in the reverse order on a given input*. We will illustrate this method in the next example.

✠ EXAMPLE 6.2-2

Find the inverse function f^{-1} for the function $y = f(x) = (2x - 5)/3$ by the *Simple Reversal Method*.

Solution

The function value $f(x) = (2x - 5)/3$ can be computed by the following series of steps:

input x ; double; subtract 5; divide by 3; output y .

The inverse action would thus be gotten by reversing both the order and the actions:

input y ; multiply by 3; add 5; halve; output x .

Thus $f^{-1}(y) = (3y + 5)/2$ must be the formula for the inverse function's action (you are asked to verify this in Exercise 25).

Perhaps you noticed in the last example that the *Simple Reversal Method* corresponds exactly to the process used to solve the given equation $y = f(x)$ for x in terms of y . According

to the *Fundamental Theorem*, this solution will yield $x = f^{-1}(y)$ as the inverse for f , if it exists. We will develop this idea further in just a moment.

But first let's point out a limitation of the *Simple Reversal Method*. If $y = f(x)$ is given by some equation but the action of f does not readily decompose into a linear sequence of consecutive operations applied to the output gotten by the preceding step, you will not be able to reverse each operation individually to determine f^{-1} , even though f may have an inverse. Illustrating this is the point of the next example.

✠ **EXAMPLE 6.2-3**

Use the *Simple Reversal Method* to find the inverse function for $f: [1, +\infty) \rightarrow [-2, +\infty)$, where $y = f(x) = x^2 - 2x - 1$.

Solution

The domain and codomain have been chosen here to make f an invertible function. Ignore this feature for the moment; we'll come back to it shortly.

The function f is most easily calculated by starting with x and squaring it to obtain x^2 , then recalling x and doubling it, next subtracting the result of step two from step one, and finally subtracting 1. Because of the branching inherent in this process (recalling x and doubling before subtracting), the inverse function cannot be found merely by reversing the steps used in calculating y .

Try it: you begin with y ; you add 1; then – what next? You run stuck.

In this case it is possible to get around the problem by recalculating the given function in another, more linear way. For instance, if you rewrite the original expression by completing the square, you will get a more complicated equivalent expression that can be calculated in serial fashion and thus reversed. This will also help you see why the domain and codomain have been chosen to be what they are. Now the *Simple Reversal Method* will succeed. Showing all this will be left as an exercise (see Exercise 30).

The difficulty met in this example points out the need for another, more systematic approach to calculating inverse functions. An ***Algebraic Solution Method*** that works in many instances also draws upon the *Fundamental Theorem of Inverse Functions*. To find the inverse g for a function f , take the equation $y = f(x)$ and solve it for x . The expression $x = g(y)$ which results will give the action of the inverse function, provided two things check out:

- 1) *The formula $x = g(y)$ defines a function with the appropriate domain and codomain.*
It is possible that the formula won't define a function: it may not give a unique output, or it may not be defined for all the values it needs to be. For example, solving $y = x^2$ for x yields $x = \pm\sqrt{y}$. Unless the domain of x -values is restricted in some way, say, to non-negative numbers, two outputs will result from a single input (or else no output will result, if $y < 0$).
- 2) *The two functions undo one another: $y = f(x) \leftrightarrow x = g(y)$.*
The forward direction automatically follows from the solution process: passing from $y = f(x)$ to $x = g(y)$ gives $y = f(x) \rightarrow x = g(y)$ via CP. If your solution process is reversible, then the other direction also holds. Otherwise, you need to check, given $x = g(y)$, that $y = f(x)$; i.e., that $y = f(g(y))$ holds.

✠ **EXAMPLE 6.2-4**

Determine the inverse function for $f(x) = \frac{x}{x+3}$. Assume that f is defined for as inclusive a set of real numbers as possible and that the codomain of f is its range.

Solution

The equation $y = \frac{x}{x+3}$ defines a function on $\mathbb{R} - \{-3\}$. We will stipulate its codomain after we determine which values y can be.

Solving $y = \frac{x}{x+3}$ for x , we get the following:

$$xy + 3y = x$$

$$xy - x = -3y$$

$$\begin{aligned}\text{Thus } x &= \frac{-3y}{y-1} \\ &= \frac{3y}{1-y} = g(y).\end{aligned}$$

Since there are x -values for all y except $y = 1$, our domain for g and our codomain for f must be taken to be $\mathbb{R} - \{1\}$.

For these x - and y -values the above solution process is reversible. The inverse function is therefore given by $f^{-1}(y) = \frac{3y}{1-y}$.

There may be times when you know that an inverse function exists on a certain domain but are unable to solve the given equation for x to find a formula for f^{-1} . When this happens, you may have to resort to inventing a new notation to stand for the inverse function and leave it at that, at least temporarily. This may sound like cheating to you, but this is exactly what is done with a number of functions you are familiar with. We will list three of these; you may be able to think of others as well.

- (1) $f(x) = x^3$; $f^{-1}(y) = \sqrt[3]{y}$.
- (2) $g(x) = 2^x$; $g^{-1}(y) = \log_2(y)$.
- (3) $h(x) = \sin x$; $h^{-1}(y) = \arcsin(y)$.

Existence of Inverse Functions

We will now take a closer look at what conditions a function must satisfy to have an inverse. In doing this, we will be developing and arguing for a theorem we have yet to state. This is not standard textbook practice, but it is typical of how theorems are discovered and proved.

We will begin by determining a necessary condition for a function f to have an inverse; that is, we will find what must hold if f is to have an inverse function g (a sort of *Method of Analysis*). We will then show that this condition is also sufficient.

Suppose, then, that $g: C \rightarrow D$ is the inverse function of $f: D \rightarrow C$.

Since g is a function from C into D , g must be defined for every y in C . But since g is an inverse of f , $x = g(y)$ iff $f(x) = y$, so f must be an onto function.

Furthermore, since g is a function, the images $x = g(y)$ are unique: exactly one x is associated with each y by g . But since g is an inverse function of f , this means there is only one x associated with a given y by f : f must be a one-to-one function.

Conjoining these two conclusions, we have the following necessary condition: *If f has an inverse function g , then f is one-to-one-and-onto.*

This condition is sufficient, too: *If f is one-to-one-and-onto, then f has an inverse function.*

Suppose f is one-to-one-and-onto, with $y = f(x)$.

Define a correspondence g from C to D such that $x = g(y)$ iff $y = f(x)$.

This correspondence is a function:

since f is onto, $g(y)$ is defined for all y ;

and since f is one-to-one, the output x is unique for any given y .

This being the case, the *Fundamental Theorem on Inverse Functions* implies that g is the inverse of f .

We have thus demonstrated the following theorem on the existence of invertible functions.

THEOREM 6.2 - 2: Existence of Invertible Functions

A function $f: D \rightarrow C$ is invertible iff f is one-to-one-and-onto.

Proof:

See above. ■

COROLLARY 1: Inverse Functions are Bijections

A function f is invertible iff f^{-1} is one-to-one-and-onto.

Proof:

This corollary claims that the necessary and sufficient condition holding for an invertible function f also holds for its inverse. See Exercise 38a. ■

COROLLARY 2: Bijections Have Bijective Inverses

A function f is one-to-one and onto iff f^{-1} is one-to-one-and-onto.

Proof:

This follows immediately from the theorem and the last corollary. See Exercise 38b. ■

COROLLARY 3: Composition of Inverses

The composition $g \circ f$ of two invertible functions f and g is invertible. Moreover, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, the composition of the inverses in the reverse order.

Proof:

See Exercise 38c. ■

EXERCISE SET 6.2

Problems 1-2: Simple Arithmetic of Functions

The following problems involve doing arithmetic operations on functions.

- *1. Let f and g be functions from \mathbb{N} to \mathbb{N} defined by $f(x) = x + 1$ and $g(x) = x^2$.
 - a. Determine a formula for the function $f + g$.
 - *b. Determine a formula for the function $f \cdot g$.
 - *c. Explain why $(f - g)(x) = f(x) - g(x)$ fails to define a function from \mathbb{N} to \mathbb{N} . Does $h(x) = |f(x) - g(x)|$ define a function from \mathbb{N} to \mathbb{N} ?
 - d. Does $h(x) = \lfloor f(x)/g(x) \rfloor$ define a function from \mathbb{N} to \mathbb{N} ? Explain.
2. Decompose the following compound functions into sums, differences, products, etc. of simpler atomic functions. Denote the basic functions involved with a letter of your choice and show how they make up the given function. Assume all functions are from \mathbb{R} to \mathbb{R} .
 - a. $h(x) = 2x^3 - x^2 + 7$
 - b. $h(x) = \sqrt{x^2 + 1}/(x - 2)$

Problems 3-9: Commuting Linear Functions

The following problems explore some simple functions and whether they commute with one another.

Let \mathcal{L} denote the set of all linear functions $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = mx + b$,

\mathcal{M} all magnifications $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = mx$, and

\mathcal{S} all shifts of the form $f: \mathbb{R} \rightarrow \mathbb{R}$ of the form $f(x) = x + b$.

Thus $\mathcal{M} \subseteq \mathcal{L}$, $\mathcal{S} \subseteq \mathcal{L}$, and $\mathcal{M} \cap \mathcal{S} = \{I_{\mathbb{R}}\}$, the singleton containing the identity function.

- *3. Show that all functions in \mathcal{M} commute with one another.
- *4. What functions in \mathcal{S} commute with all functions in \mathcal{M} ? Justify your answer.
- *5. What functions in \mathcal{L} commute with all functions in \mathcal{M} ? Justify your answer.
6. Show that all functions in \mathcal{S} commute with one another.

7. What functions in \mathcal{M} commute with all functions in \mathcal{S} ? Justify your answer.
8. What functions in \mathcal{L} commute with all functions in \mathcal{S} ? Justify your answer.
9. What functions in \mathcal{L} commute with all functions in \mathcal{L} ? Justify your answer.

Problems 10-11: Non-Commutativity and Associativity of Composition

Work the following problems related to algebraic properties of composition.

10. Choose two different non-linear real-valued functions of a real variable and determine whether or not they commute.
11. Verify that composition is associative, as *Proposition 2* asserts, in the case where $f(x) = 2x - 1$, $g(x) = 3 - x$, and $h(x) = x^2 + 1$: calculate the two different composite functions in stages and compare the final results.

Problems 12-13: Composition of One-to-One and Onto Functions

The following problems look at how composition interacts with functions being one-to-one or onto.

- *12. *Proposition 3*
Prove *Proposition 3b*: If f and g are both onto functions, then so is $g \circ f$.
- *13. *Converses and Partial Converses to Proposition 3*
 - *a. Show that the following full converse to *Proposition 3a* is false: If $g \circ f$ is one-to-one, then so are both f and g .
 - *b. Prove the following partial converse of *Proposition 3a*: If $g \circ f$ is one-to-one, then so is f .
 - c. Show that the full converse to *Proposition 3b* is false: If $g \circ f$ is onto, then so are both f and g .
 - d. Prove the following partial converse of *Proposition 3b*: If $g \circ f$ is onto, then so is g .
 - e. Show that the full converse to *Proposition 3c* is false: If $g \circ f$ is a one-to-one onto function, then so are both f and g .
 - f. Formulate and prove a partial converse to *Proposition 3c* (see parts *b* and *d*).
 - *g. True or false: f is a one-to-one/onto function iff $f \circ f$ is a one-to-one/onto function. Explain, using *Proposition 3* along with the results of parts *b* and *d*. Assume $f \circ f$ is defined.

Problems 14-15: True or False

Are the following statements true or false? Explain your answer.

14. The *Simple Reversal Method* can always be used to find the inverse of an invertible function.
- *15. For invertible functions f and g , $(f \circ g)^{-1} = f^{-1} \circ g^{-1}$.

Problems 16-18: Identities

The following problems deal with identity functions and their properties.

16. *Uniqueness of Identity Elements*
Let \mathcal{F} denote a collection of functions. Prove that \mathcal{F} contains at most one identity element i for function composition.
17. *Identity Functions*
Show that the identity function $I_S : S \rightarrow S$ is one-to-one-and-onto.
18. *Proposition 4*
Prove *Proposition 4*: The identity function I_S is the identity element for composition in the collection S^S of all functions $f : S \rightarrow S$.

Problems 19-24: Function Exponents

The following problems give a number of results having to do with function exponents. These are defined recursively for all natural numbers and then extended to all integers as follows:

Definition: Let f be a function from a set S into itself.

$$f^0 = I_S; \quad f^{n+1} = f^n \circ f \text{ for all } n \text{ in } \mathbb{N}.$$

$$f^{-n} = (f^{-1})^n \text{ for all } n \text{ in } \mathbb{N}^+, \text{ provided } f \text{ is invertible.}$$

- *19. Prove that $f^m \circ f^n = f^{m+n}$ for all m and n in \mathbb{N} .

20. Prove that $(f^m)^n = f^{mn}$ for all m and n in \mathbb{N} .
21. Prove that $(f \circ g)^n = f^n \circ g^n$ for all n in \mathbb{N} iff $f \circ g = g \circ f$.
- *22. Prove that $f^{-n} = (f^n)^{-1}$ for all n in \mathbb{N} .
23. Prove $f^m \circ f^n = f^{m+n}$ for integer exponents m and n .
24. Prove $(f^m)^n = f^{mn}$ for integer exponents m and n .

Problems 25-38: Inverse Functions

The following problems deal with inverse functions and their properties.

- *25. Using the definition of inverse functions, show that the function $g(y) = (3y + 5)/2$, determined in Example 2, is the inverse of $f(x) = (2x - 5)/3$.
26. *Proposition 5*
 Prove the alternative characterization of inverse functions given in Proposition 5: If $f: D \rightarrow C$ and $g: C \rightarrow D$, then f and g are inverses iff $g(f(x)) = x$ for all x in D and $f(g(y)) = y$ for all y in C .
27. *Inverses of Inverses*
 Using Definition 4 or Proposition 5, prove that if f is invertible, so is its inverse f^{-1} , with $(f^{-1})^{-1} = f$.
28. *Distinguishing Inverses for Composition and Multiplication*
 a. Find a real-valued function f such that $f^{-1}(x) \neq 1/f(x)$ (this is the normal situation).
 b. Exploration: determine whether any real-valued functions f satisfy $f^{-1}(x) = 1/f(x)$ (a highly unusual situation).
- *29. Use the *Simple Reversal Method* to determine the inverse function for $y = f(x) = (2 - x)/3$. Check that the result you get is the inverse by means of Proposition 5.
- *30. To finish Example 3, calculate the inverse for $f: [1, +\infty) \rightarrow [-2, +\infty)$ defined by $y = f(x) = x^2 - 2x - 1$ by rewriting $f(x)$ using the process of completing the square. Then analyze the computation process this formula exhibits, and reverse the steps to get the inverse. Show that the function you get is the inverse function f^{-1} by means of Proposition 5. In the process, explain why the domain and codomain are chosen the way they are.
31. Calculate the inverse for $f: [1, +\infty) \rightarrow [-2, +\infty)$ defined by $y = f(x) = x^2 - 2x - 1$ by solving for $x = g(y)$ using the quadratic formula. Compare your result with what was found in Exercise 30.
Caution: you are to solve for x given any y , not find a zero for the function (when $y = 0$).
- *32. Show that $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x - 5$ is both one-to-one and onto and thus is invertible. Find its inverse $x = f^{-1}(y)$ and explain why it must be the inverse of f .
33. Show that $f: [1, +\infty) \rightarrow [0, +\infty)$ defined by $f(x) = x^2 - 2x + 1$ is invertible and find its inverse.
- *34. Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3 + 2$ is invertible and find its inverse.
35. Show that the function $f: \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x^3 + x$ is invertible and find its inverse.
- *36. Show that the function f whose action is given by $f(x) = \frac{2-x}{x+1}$ and whose domain is $\mathbb{R} - \{-1\}$ is a one-to-one function onto its range. Then calculate its inverse $x = f^{-1}(y)$ and specify both its domain and range.
37. Show that $f(x) = x/(1-x^2)$ is an invertible function from the open interval $(-1, +1)$ into \mathbb{R} by showing it is one-to-one-and-onto. Then find its inverse $x = f^{-1}(y)$. [Hint: Make use of the quadratic formula.]
38. *Corollaries to Theorem 2*
 a. Prove Corollary 1 to Theorem 2: A function f is invertible iff f^{-1} is one-to-one-and-onto.
 b. Prove Corollary 2 to Theorem 2: A function f is one-to-one-and-onto iff f^{-1} is one-to-one-and-onto.
 c. Prove Corollary 3 to Theorem 2: The composition $g \circ f$ of two invertible functions f and g is invertible. Moreover, $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$, the composition of the inverses in the reverse order.

Problems 39-41: Exploring Identities and Inverses

The following problems explore the concepts of one-sided identities and inverses.

39. *Exploring Identities for a Function and the Identity Function*

A function $g: S \rightarrow S$ is a *left identity* for a function $f: S \rightarrow S$ iff $g \circ f = f$.

A function $g: S \rightarrow S$ is a *right identity* for a function $f: S \rightarrow S$ iff $f \circ g = f$.

A function $g: S \rightarrow S$ is a *two-sided identity* for a function $f: S \rightarrow S$ iff g is both a left and a right identity for f .

- Show that I_S is both a left identity and a right identity for each function $f: S \rightarrow S$.
- Do left and right identities for a function need to be unique? Or could there be two functions that act like left identities (respectively, right identities) for a given function f ? Support your answer.
- Do two-sided identities for a function need to be unique? Support your answer.
- Does your answer to part *c* conflict with the claim in the text that composition identities are unique? Explain.

40. *Exploring Inverses for a Function and Inverse Functions*

- Using Exercise 39 as your model, define left and right and two-sided inverses for a function.
- Do left and right inverses for a function have to be unique? Can a function have a left inverse without having a right inverse, or conversely?
- Can a function have a left inverse and a right inverse, but have no two-sided inverse? If a function has a two-sided inverse, does it need to be unique?

41. *Inverses, Cancellation, and Composition*

- Show that $f: D \rightarrow C$ is one-to-one iff f has a left inverse $g: C \rightarrow D$; i.e., iff $g \circ f = I_D$.
- Show that if $f: D \rightarrow C$ is one-to-one, then it can be canceled on the left: $f \circ g_1 = f \circ g_2 \rightarrow g_1 = g_2$.
- Show that $f: D \rightarrow C$ is onto iff f has a right inverse $g: C \rightarrow D$; i.e., iff $f \circ g = I_C$.
- Show that if $f: D \rightarrow C$ is onto, then it can be canceled on the right: $g_1 \circ f = g_2 \circ f \rightarrow g_1 = g_2$.
- Show that $f: D \rightarrow C$ is one-to-one-and-onto iff it can be canceled on the left and on the right; i.e., iff $f \circ g_1 = f \circ g_2 \rightarrow g_1 = g_2$ and $g_1 \circ f = g_2 \circ f \rightarrow g_1 = g_2$.

Problems 42-43: Composition and Inverses of Images and Pre-Images,

The following problems relate composition and invertibility to set images and set pre-images.

Let $f: D \rightarrow C$, $g: C \rightarrow B$, $U \subseteq D$, $V \subseteq C$, and $W \subseteq B$.

42. *Composition of Set Images and Pre-Images*

Prove the following.

- $(g \circ f)[U] = g[f[U]]$
- $(g \circ f)^*[W] = f^*[g^*[W]]$

EC 43. *Set Pre-Images, Set Images, and Invertible Functions*

Prove the following, where $f: D \rightarrow C$ denotes an invertible function.

- $f^*[V] = f^{-1}[V]$
- $(f^{-1})^*[U] = f[U]$

HINTS TO STARRED EXERCISES 6.2

1. b. See Definition 1.
c. What could go wrong with subtraction?
3. Take two functions $f_1(x) = m_1x$ and $f_2(x) = m_2x$ and show that $f_1 \circ f_2 = f_2 \circ f_1$.
4. Take two functions $f_1(x) = x + b$ and $f_2(x) = mx$ and see for which functions $f_1 \circ f_2 = f_2 \circ f_1$.
5. Follow a procedure similar to that of Problems 1 and 2.
12. Begin with an element $z \in B$. Find an element $x \in D$ so that $g \circ f(x) = z$. Use the fact that both f and g are onto functions.
13. a. Find two functions f and g defined on sets of your choice so that the composite function $g \circ f$ is one-to-one but at least one of the two component functions is not. See part *b* for a necessary condition, however.
b. Use the definition of being one-to-one plus your knowledge of logic to design a proof.
15. [No hint.]
19. Prove this via induction on n .
22. Since inverses are unique, show that f^{-n} is the inverse of f^n via the definition, using the result of Problem 17 where appropriate. No induction is needed here.
25. Follow the procedure of Example 1.
29. Follow the procedure of Example 2.
30. Follow the instructions given in the problem.
32. This has a number of parts, none of them difficult. Take it a piece at a time.
34. To show that f is invertible, use *Theorem 2*: show it is one-to-one and onto. Find f^{-1} using any method.
36. Use the definitions to show that f has the requisite properties. Calculate the inverse via the *Algebraic Solution Method* and explain why it is f^{-1} .

6.3 Equivalence Relations and Partitions

A topic that *Set Theory* brings much greater deductive precision to than it previously had is the *theory of relations*. In the final analysis this is due to the Wiener-Kuratowski definition of ordered pairs (see Exercise Set 4.3), but here we will use our informal notion of an ordered pair, which doesn't assume that everything needs to be reduced to sets. That already suffices to put relations on a set-theoretic foundation of sorts.

Functions are a special kind of binary relation, so the topic in this section generalizes those of Sections 6.1 and 6.2. Other important types of relations are equivalence relations and various ordering relations. Here we will focus on binary relations in general and equivalence relations in particular. Equivalence relations come up often in mathematics. We have already seen several such relations, and we will look at additional significant examples here. Other relations will be taken up in Chapter 7, where they will be studied in a more algebraic setting.

Definition of Binary Relations and Relations in General

A binary relation holding between two sets of objects involves a particular way of pairing them off. Such relations have a definite directionality. We say, for example, that one person *is the father of* another. This relation cannot be turned around; the child is not the father of the man, Wordsworth notwithstanding. "Is the father of" is a relationship realized by many pairs of individuals, by your father and you as well as by my father and me and by me and my children.

If we focus on the pairs of things being related, we can give an extensional characterization of that relation. A binary relation is modeled by the set of all ordered pairs of objects having the given relationship. Using this approach, relations can be treated in a precise, set-theoretical manner, as the following definition specifies.

DEFINITION 6.3-1: Binary Relation (set-theoretical version)

- R is a **binary relation between sets S and T** (or from S to T) iff $R \subseteq S \times T$.
- R is a **relation on S** iff R is a relation from S to S ; i.e., iff $R \subseteq S \times S$.
- R is a **binary relation** iff R is a binary relation between some sets S and T .

✂ EXAMPLE 6.3-1

Give two examples of binary relations.

Solution

(a) The relation "_____ is a student enrolled in _____" is a binary relation holding between students and courses offered. According to the definition, we can think of this relation as the set of all pairs (x, y) , where x is a student taking course y .

(b) The relation "_____ is less than _____" is a relation on the set \mathbb{R} of all real numbers. Technically, following Definition 1 and using what we know from *Set Theory*, the official way to indicate that $3 < \pi$ is by writing $(3, \pi) \in <$. Frankly, this looks pretty weird.* We will mostly use conventional *infix notation* ($3 < \pi$), but occasionally the more formal set-theoretical notation will be useful.

Thinking of relations as sets of ordered pairs, as in the definition, is admittedly awkward. It is really only done in order to treat relations as a part of *Set Theory*. Mathematicians who are not set theorists usually think of a binary relation as a certain connection between pairs of things, not as the set of ordered pairs of related objects. This practice is reflected

* Moreover, the relation of set membership is necessarily still treated intentionally; how would one write that $((3, \pi), <)$ is in \in without entering a vicious circle, once again using \in or some symbol meaning essentially the same thing?

in standard mathematical notation, as we noted. We ordinarily write $\triangle ABC \sim \triangle A'B'C'$, not $(\triangle ABC, \triangle A'B'C') \in \sim$. A claim that x has relation R to y is invariably written as $x R y$, not as $(x, y) \in R$. To make use of this convenient notation and thinking even while maintaining/permitting a set-theoretical perspective, we can adopt the following definition.

DEFINITION 6.3-2: Notation for Binary Relations

Let R be a binary relation. Then $x R y \leftrightarrow (x, y) \in R$.

To deal with binary relations, it is probably good to think about them in a dual way. Informally, a relation R is the way in which the elements of the two sets are associated. As such, it is an intangible linkage that transcends the ordered pairs of elements being related. Formally, though, R can be thought of as the set of all ordered pairs (x, y) for which x is related to y in the informal sense of the relation.

Strange as the extensional definition of a binary relation may seem, just as in the case of the set-theoretical definition of ordered pair or the set-theoretical treatment of natural numbers (see Exercise Set 5.3), its value lies in what can be done with it inside *Set Theory*. Given this definition, various general theorems about relations can be rigorously proved. This is usually taken as adequate justification for its introduction, just as hypotheses in physics or any other science are thought to be justified by means of *their* consequences.

Associated with each binary relation R there are two sets called the *domain* and the *range* of the relation. These are, respectively, the set of all first elements of the relation and the set of all second elements. As a binary relation, $R \subseteq S \times T$ for some sets S and T . These sets may themselves be the domain and range, but nothing requires them to be so. All we can say for sure is that the domain of R is a subset of S and the range of R is a subset of T .

DEFINITION 6.3-3: Domain and Range of a Binary Relation

Suppose R is a binary relation, with $R \subseteq S \times T$. Then

- a) The **domain** of R is $\text{Dom}(R) = \{x \in S : x R y \text{ for some } y \in T\}$
- b) The **range** of R is $\text{Rng}(R) = \{y \in T : x R y \text{ for some } x \in S\}$.

✠ **EXAMPLE 6.3-2**

Show that both functions and operations on sets can be considered relations and specify their domain and range.

Solution

- Let $F: S \rightarrow T$ be a function. F relates its inputs to their outputs and so can be thought of as subset of $S \times T$. F is a binary relation that satisfies a special condition (uniqueness of outputs). In place of $y = F(x)$, therefore, we might write either $x F y$ or $(x, y) \in F$ in order to make use of binary relation notation. The notions of domain and range for F as a function agree with those given for treating it as a relation.
- To see how to think about binary operations, consider an operation such as addition on \mathbb{Z} . Given any two integers a and b , there exists a unique integer c such that $a + b = c$. Thus, we can think about addition as a function $+: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$, with $+(a, b) = c$. Thus, by what we just said about functions, $+$ can be considered a binary relation between ordered pairs of integers (domain: $\mathbb{Z} \times \mathbb{Z}$) and integer outputs (range: \mathbb{Z}). Alternatively, addition can be considered a ternary relation comprised of the appropriate ordered triples (a, b, c) where $a + b = c$. All binary operations can be treated similarly as functions and relations.

Our focus in this text is on binary relations, but as the last example begins to point out, many relations connect more than just two things at a time. In fact, computer scientists who design data structures are interested in n -ary relations; that is, in relations holding among n different things. These provide relational data models for representing and manipulating

information in a database. Any table containing n columns of data can be thought of as an n -ary relation. Such relations can be modeled and worked with using ordered n -tuples and n -fold Cartesian products.

DEFINITION: N -ary Relations

R is an **n -ary relation** iff R is a set of ordered n -tuples; i.e., iff $R \subseteq S_1 \times S_2 \times \cdots \times S_n$ for some sets S_1, S_2, \dots, S_n .

Types of Binary Relations and Their Properties

A binary relation R may be from a set S to a different set T , or it may be from a set S to itself. Relations of the first type are important, as we have seen, because they give us a way to handle functions and operations. But relations of the second type are also important because they allow us to treat order and conceptualize relational structure within a single set.

For the rest of this section we will focus on relations of this second type. We will begin by presenting various properties such relations may have. Certain key relations have three important properties and are called *equivalence relations*. Others have two or three other properties and are called partial, total, or well orderings; we will pick these up later (see Section 7.1). *Note that it is a relation as a whole that has or doesn't have these properties.*

We have mentioned equivalence relations several times earlier in the text, defining their properties as the occasion arose. We will now (re)state the relevant definitions and illustrate them. We will also define some closely related binary relations.

DEFINITION 6.3-4: Reflexive Relations

A relation R on a set S is **reflexive** iff $(\forall x \in S)(x R x)$.

DEFINITION 6.3-5: Identity Relation on S

The **identity relation** D on a set S is the relation defined by $x D y$ iff $x = y$.

Note that in order for a relation R to be reflexive, the pair (x, x) must be in R for all x in S ; everything must be related to itself by the relation R . Graphically, this means that the entire main diagonal D of $S \times S$ must be in (the graph of) the relation R . Set theoretically, R is reflexive iff $D \subseteq R$, where D is the identity relation on S . Other pairs/points may also be in R , but they need not be in order for R to be reflexive, nor do they affect whether or not R is reflexive (see Exercise 37).

DEFINITION 6.3-6: Symmetric Relations

A relation R on a set S is **symmetric** iff $(\forall x, y \in S)(x R y \rightarrow y R x)$.

In terms of its graph as a subset of $S \times S$, a symmetric relation R has the following property: whenever (x, y) is in (the graph of) R , then its mirror image (y, x) across the diagonal must also be in (the graph of) R .

DEFINITION 6.3-7: Converse Relation

If R is a relation from S to T , the **converse relation** of R is $\widehat{R} = \{(x, y) : (y, x) \in R\}$.

Converse relations can be thought of, more or less, as being the same relationship viewed in reverse. For example, the order-relation $<$ on the set \mathbb{R} is the converse of the relation $>$: $a < b$ iff $b > a$.

Converse relations are important to us here because we can use them to characterize symmetric relations. For if two elements of S are related by a symmetric relation, they must be related in both directions. In set-theoretic terminology, R is a symmetric relation iff it is equal to its converse: $\widehat{\widehat{R}} = R$.

DEFINITION 6.3-8: Transitive Relations

A relation R on a set S is **transitive** iff $(\forall x, y, z \in S)(x R y \wedge y R z \rightarrow x R z)$.

Transitive relations have a certain degree of linkage or connectivity. If an element x is related to some element y , which in turn is related to an element z , then x must be related to z , too. Ordinary $<$ on \mathbb{R} is an example of such a relation.

DEFINITION 6.3-9: Composite Relation

Suppose R_1 is a relation from S to T and R_2 is a relation from T to U . The **composite relation** $R = R_1 \circ R_2$ is the binary relation from S to U defined by $R = \{(x, z) : x R_1 y \text{ and } y R_2 z \text{ for some } y \in T\}$.

[Note the order in which composite relations are symbolized. Unfortunately, this is the reverse of how composite functions are written, though they are also composite relations. Our infix notation for relations is responsible for this; the first relation is put next to the element of the first set: $(x, z) \in R_1 \circ R_2$ iff $x (R_1 \circ R_2) z$. For functions, on the other hand, standard notation puts the first element (the function's input) on the right: $z = (f_2 \circ f_1)(x)$.]

Composite relations are formed from two simpler relations and usually lead to a new relation, even if the two given relations are the same. For example, the relation “is the father of” composed with itself becomes the composite relation “is the grandfather of”. However, there are times when the composite of a relation with itself gives the original relation. In fact, this is roughly what characterizes a transitive relation: *a relation R on a set S is transitive iff $R^2 \subseteq R$, where $R^2 = R \circ R$ (see Exercise 80).*

✠ EXAMPLE 6.3-3

Exhibit several transitive relations from earlier in the book, and determine whether they are also reflexive or symmetric.

Solution

- We first met a transitive relation in Section 1.3 when we discussed the relation of logical equivalence holding among propositions. If $P \vDash Q$ and $Q \vDash R$, then $P \vDash R$. This relation is also reflexive ($P \vDash P$ for all P) and symmetric (if $P \vDash Q$, then $Q \vDash P$). Thus, true to its name, logical equivalence is an equivalence relation (see Definition 10 below). The same is true of interderivability (\vDash): it is likewise an equivalence relation.
- The relations of logical implication (\vDash) and derivability (\vdash) are transitive and reflexive, but they are not symmetric.
- The relation $<$ is a transitive relation on the set of natural numbers \mathbb{N} , but it is neither reflexive nor symmetric. The relation \leq is reflexive as well as transitive, but it is still not symmetric.
- The relation of divisibility on the set of integers \mathbb{Z} , denoted by $x \mid y$, is a transitive relation. It is also a reflexive but not symmetric. The converse of “divides” is “is a multiple of”; these relations do not coincide.

DEFINITION 6.3-10: Equivalence Relation

R is an **equivalence relation** on S iff R is reflexive, symmetric, and transitive.

As just noted, we've encountered equivalence relations on several occasions already: logical equivalence, interderivability, and identity are equivalence relations from *Logic*; set equality and equinumerosity are two more from *Set Theory*. Equivalence relations are very special relations, sharing three key features with identity; in fact, equivalence relations can often be considered a generalization of identity in that the objects they relate are identical with respect to some property or other (amount, truth-value, shape, etc.). Equivalence relations appear in all branches of mathematics and underly a very important set-theoretical construction, generating what is called a quotient structure of equivalence classes (see below).

✧ **EXAMPLE 6.3 - 4**

Exhibit several additional equivalence relations from different branches of mathematics.

Solution

- a) The relation P defined by $x P y \leftrightarrow 2 \mid (x - y)$, is an equivalence relation on the integers. It is easy to show that P is reflexive, symmetric, and transitive. If $x P y$, then either x and y are both even or they are both odd (see Exercise 49). When this holds, we say x and y have the same parity. Thus, P is the relation *has the same parity as*.
- b) Generalizing, the relation defined by $x R y \leftrightarrow n \mid (x - y)$ is also an equivalence relation (see Exercise 50). Since $n \mid (x - y)$ iff x and y each yield the same remainder upon division by n (this generalizes being even or odd; see Exercise 51), $x R y$ iff x and y give the same remainder when divided by n . This equivalence relation is called *congruence modulo n* in number theory and forms the basis of modular arithmetic (see Example 7).
- c) The relations of congruence and similarity in geometry are equivalence relations. For, every figure is congruent/similar to itself; if two figures are congruent/similar, they are so in either order; and if one figure is congruent/similar to two others, those others are congruent/similar to one another.
- d) In calculus the relation *has the same derivative as* is an equivalence relation among differentiable functions. Every differentiable function has a derivative, which is unique, so the relation is reflexive; if two functions have the same derivative, they do so regardless of which derivative is taken first (the relation is symmetric); and if one function has the same derivative as a second, and the second has the same derivative as the third, then the first and third functions have the same derivative (the relation is transitive). This relation is shown in an introductory calculus course to be identical with the relation *differs by a constant from* defined on the same set of differentiable functions: two differentiable functions have the same derivative iff they differ by a constant. This result provides the basis for antidifferentiation or indefinite integration.

Equivalence Relations, Partitions, and Quotient Structures

One of the reasons equivalent relations are so pervasive and significant in mathematics is because they are associated with partitions of sets. Such partitions provide a new structure of objects that can often be worked with in roughly the same way as the original set. We will look at this important connection here and give some examples.

Given an arbitrary equivalence relation \sim defined on a set S , we know that every element of the set is related to something because it is at least related to itself (reflexivity). This may be the only element related to it, or it may be related to others. If you collect together all the elements that are related to one another, you will see a pattern start to emerge. We will illustrate this first by means of some examples and then generalize them to state a theorem. To help simplify our treatment, though, we will first introduce some terminology and notation.

DEFINITION 6.3 - 11: Equivalence Class

Let \sim be an equivalence relation on a set S and let $x \in S$. The **equivalence class** of x , denoted by $[x]$, is the set $[x] = \{y \in S : x \sim y\}$.

✧ **EXAMPLE 6.3 - 5**

Give some examples of sets having equivalence relations defined on them and identify their equivalence-class structure.

Solution

- a) Take \mathbb{Z} as our set and define $n \sim m$ iff both numbers have the same (non-negative) remainder when divided by 3.
Since each integer has a unique remainder and since there are only three permissible

remainders, 0, 1, and 2, each integer will belong to one and only one equivalence class: $[0]$, $[1]$, or $[2]$. For example, $8 \in [2]$ because $8 = 3(2) + \underline{2}$, $-6 \in [0]$ because $-6 = 3(-2) + \underline{0}$, and $-2 \in [1]$ because $-2 = 3(-1) + \underline{1}$.

- b) Take S to be the set of all equilateral triangles in a given plane and let \sim denote the relation *has the same area as*, which is an equivalence relation. The equivalence classes here turn out to be sets of congruent triangles, one for each size triangle. We could therefore index this collection of equivalence classes by means of positive real numbers (length of the side). These classes jointly contain all equilateral triangles without any membership overlap between the classes.
- c) Our third example comes from algebra. Let S be the set of all polynomials with real coefficients, and take \sim to be the relation *is a non-zero real multiple of*. This relation is an equivalence relation (see Exercise 74); the equivalence classes contain all polynomials of the same degree that are related to one another as non-zero scalar multiples. All polynomials, including the zero polynomial, are in some equivalence class, and none of them belongs to two different classes. All equivalent classes here are infinite except for $[0]$, which contains only the zero polynomial.
- d) Finally, suppose $f: S \rightarrow T$ is a function, and define $a \sim b$ iff $f(a) = f(b)$. Then \sim is an equivalence relation on S (see Exercise 81a). The equivalence classes are subsets of the domain whose elements have a common image: $[a] = \{b \in S: f(a) = f(b)\}$.

As should be apparent from these four examples, an equivalence relation completely carves up a given set S into mutually exclusive equivalence classes. In more technical terms, the equivalence classes form a *partition* of S . We defined this concept earlier (Section 4.2) but will repeat it here for easy reference.

DEFINITION 6.3-12: Partition of a Set

A collection \mathcal{P} of non-empty sets is a partition of a set S iff \mathcal{P} is pairwise disjoint and $\bigcup_{T \in \mathcal{P}} T = S$. Each member of \mathcal{P} is called a cell of the partition.

✠ **EXAMPLE 6.3-6**

Determine which of the following collections \mathcal{P} are partitions of the given set S .

- a) $S = \mathbb{Z}$; \mathcal{P} consists of the three sets $R_0 = \{n \in \mathbb{Z}: n = 3k\}$, $R_1 = \{n \in \mathbb{Z}: n = 3k + 1\}$, and $R_2 = \{n \in \mathbb{Z}: n = 3k + 2\}$.
- b) $S = \mathbb{N}$; \mathcal{P} consists of the two sets P and C , where P is the set of prime numbers, $\{2, 3, 5, 7, \dots\}$, and C is the set of composite numbers, $\{4, 6, 8, 9, \dots\}$.
- c) $S = \mathbb{Z}$; \mathcal{P} is the collection of all sets $k\mathbb{Z} = \{n \in \mathbb{Z}: n = mk \text{ for some } m \in \mathbb{Z}\}$ for each $k \in \mathbb{N}$; $3\mathbb{Z} = \{0, \pm 3, \pm 6, \pm 9, \dots\}$, for example.

Solution

- a) \mathcal{P} is a partition of \mathbb{Z} because it is a collection of sets that all together exhaust \mathbb{Z} and that are pairwise disjoint. Given an integer, its remainder is unique when divided by 3, so each integer belongs to exactly one cell of the collection.
Note that this partition is the collection of equivalence classes given in Example 5a above. We can thus use this partition to define that equivalence relation \sim on \mathbb{Z} : two elements x and y are related by \sim iff they belong to the same cell of this partition \mathcal{P} .
- b) \mathcal{P} is not a partition of \mathbb{N} in this case, for while P and C are disjoint, they do not exhaust \mathbb{N} ; neither 0 nor 1 are either prime or composite.
- c) \mathcal{P} also is not a partition here, because while the sets $k\mathbb{Z}$ jointly exhaust \mathbb{Z} , they are not pairwise disjoint; in particular, every set contains the number 0.

Generalizing Examples 5 and 6, we can claim that an equivalence relation induces a partition of the set, and conversely. These two assertions are the content of the next theorem.

THEOREM 6.3-1: Fundamental Theorem of Equivalence Relations

- a) If \sim is an equivalence relation on S , then its equivalence classes form a partition of S .
- b) If \mathcal{P} is a partition of S , then \mathcal{P} generates an equivalence relation on S ; moreover, the equivalence classes so created are the cells of \mathcal{P} .

Proof:

- a) Suppose that \sim is an equivalence relation on a set S , and let $\mathcal{P}_\sim = \{[x] : x \in S\}$.
[We will show that \mathcal{P}_\sim is a partition.]
Each x is in $[x]$ because $x \sim x$ for equivalence relations. Moreover, $[x] \subseteq S$ for each x .
Thus the total union $\bigcup_{[x] \in \mathcal{P}_\sim} [x]$ of all equivalence classes gives S .
[It remains for us to show that \mathcal{P}_\sim is a pairwise-disjoint collection.]
Let $[a], [b]$ be any two equivalence classes.
[We must prove: either $[a] = [b]$ or $[a] \cap [b] = \emptyset$; i.e., classes have all or nothing in common.
Caution: a and b can be different and still give the same equivalence class.]
Suppose $[a] \cap [b] \neq \emptyset$.
[Using *EO*, our proof will be complete if we can prove $[a] = [b]$. We do this by showing they are equal sets: $x \in [a] \leftrightarrow x \in [b]$.]
First suppose $x \in [a]$. Then $a \sim x$.
[We want to show that $x \in [b]$, which is done by showing that $b \sim x$. We do this by finding a connecting element relating x and b .]
Since $[a] \cap [b] \neq \emptyset$, let $c \in [a] \cap [b]$.
Then $a \sim c$ and $b \sim c$.
By symmetry, $c \sim a$.
Thus, since $b \sim c \sim a \sim x$, transitivity yields $b \sim x$.
This shows that $x \in [a] \rightarrow x \in [b]$.
The other direction follows in exactly the same way: interchange a and b . [To be sure you followed the argument just given, try to prove this on your own (see Exercise 79a).]
- b) Now suppose that \mathcal{P} is any partition of S .
[We wish to define an equivalence relation $\sim_{\mathcal{P}}$ on S so that the resulting collection of equivalence classes turns out to be the original partition. Knowing that elements in the same equivalence class must be related, we use our partition cells to define the relation to make that true.]
Define $\sim_{\mathcal{P}}$ by $x \sim_{\mathcal{P}} y$ iff x and y belong to the same cell of the partition.
It is not difficult to show that $\sim_{\mathcal{P}}$ is an equivalence relation (see Exercise 79c).
[To prove that the collection of equivalence classes associated with $\sim_{\mathcal{P}}$ is the original partition \mathcal{P} , we must show that every equivalence class is a cell of the partition, and conversely.]
Let $[a]$ be any equivalence class.
Given any $x \in [a]$, $a \sim_{\mathcal{P}} x$.
Thus a and x are both in some cell C of the partition, which means $[a] \subseteq C$.
Given any $z \in C$, z is in the same cell of the partition as a , too, so $a \sim_{\mathcal{P}} z$.
Thus $z \in [a]$, and so $C \subseteq [a]$.
Hence $[a] = C$: each equivalence class is a member of the partition.
Now let C denote an arbitrary member of the partition.
Let $x \in C$ (all cells are non-empty). [We show that $C = [x]$.]
If $z \in C$, then $x \sim_{\mathcal{P}} z$, and so $z \in [x]$.
If $y \in [x]$, then $x \sim_{\mathcal{P}} y$.
But then x and y belong to the same member of the partition, which must be C .
Hence $y \in C$.
Thus $C = [x]$.
Therefore the partition of equivalence classes induced by the equivalence relation $\sim_{\mathcal{P}}$ is the same as the original partition. ■

The most important half of this theorem is the forward direction: *equivalence classes form a partition*. Thus, once you have an equivalence relation, you can break the original set up into equivalence classes. These cells form a new set of objects, a mathematical structure known as a *quotient structure*, that can then be worked with further. In many instances, this new collection (the partition) has or can be given the same type of structure as the original set.

✠ **EXAMPLE 6.3-7**

Describe the equivalence classes of the relation \equiv_n (*congruence mod n*) defined on \mathbb{Z} by $a \equiv_n b$ iff $n \mid (a - b)$.

Solution

We first note that \equiv_n is an equivalence relation: see Example 4b and Exercise 50.

In the second place, $a \equiv_n b$ iff a and b have the same remainder when divided by n (see Exercise 51).

Thus, an equivalence class for \equiv_n is a subset of integers that all have the same remainder. This gives n distinct residue classes $[0], [1], \dots, [n-1]$, one for each remainder; $x \in [i]$ iff the remainder left when x is divided by n is i .

These classes can be treated as single objects and an arithmetic designed for them. For instance, a class addition can be defined on the partition by $[a] \oplus [b] = [a + b]$. We will not go into the details here, but this is essentially mod n addition or “clock” arithmetic for n hours. This arithmetic shares many properties with ordinary arithmetic. Modular arithmetic has a host of applications; you will probably meet up with it in other mathematics or computer science classes.

EXERCISE SET 6.3

Problems 1-8: Binary Relations

Which of the following sets represent binary relations? For those that do, identify the domain and range of the relation and graph the relation. For those that do not, explain why they aren't binary relations.

- *1. $\{(0, 0), (0, 1), (1, 0), (1, 1)\}$
- *2. $\{0, (0, 0), (0, 1), (1, 0), (1, 1), 1\}$
- 3. $\{(\emptyset, \emptyset), (\emptyset, \{\emptyset\}), (\{\emptyset\}, \emptyset), (\{\emptyset\}, \{\emptyset\})\}$
- 4. $S \times T$, where $S = \{0, 1, 2\}$, $T = \{2, 3, 4, 5\}$
- 5. \mathbb{N}
- *6. $R = \{(x, y) \in \mathbb{R}^2 : x + y \leq 1\}$
- 7. $R = \{(x, y) \in \mathbb{R}^2 : x^2 < y^2\}$
- 8. $\{(x, y, z) \in \mathbb{N}^3 : x^2 + y^2 = z^2\}$

*9. *Counting Relations*

Suppose S is a set with $|S| = n$. Use the set-theoretic definition of binary relation and your knowledge of combinatorics to answer the following.

- *a. How many different binary relations are there on S ?
- b. How many different binary relations on S are reflexive?
- c. How many different binary relations on S are symmetric?

EC

Problems 10-13: True or False

Are the following statements true or false? Explain your answer.

- *10. All equivalence relations are transitive relations.

11. Ordinary \leq is a symmetric relation.
12. Two equivalence classes $[x]$ and $[y]$ are identical iff $x = y$.
13. Any binary relation R can be represented set theoretically as $\text{Dom}(R) \times \text{Rng}(R)$.

Problems 14-17: Relations and Operations of Set Theory

The following problems explore how relations connect with certain relations and operations of Set Theory.

14. Prove that if R is a relation and $S \subseteq R$, then S is a relation.
15. Prove that if R and S are both n -ary relations, then so are the following:
 - a. $R \cap S$
 - b. $R - S$
 - c. $R \cup S$
16. Prove that $S \times T$ is a binary relation for any two sets S and T .
17. Prove or disprove: any binary relation R can be written as a Cartesian product of two sets.

Problems 18-23: Domains, Ranges, and Set Theory

Prove or disprove the following results dealing with the interaction of the domain and range of binary relations R and S with the operations of Set Theory. If a result is false, see whether something can be done to "fix" it, and then prove that result.

18. $\text{Dom}(R \cap S) = \text{Dom}(R) \cap \text{Dom}(S)$
19. $\text{Dom}(R - S) = \text{Dom}(R) - \text{Dom}(S)$
20. $\text{Dom}(R \cup S) = \text{Dom}(R) \cup \text{Dom}(S)$
21. $\text{Rng}(R \cap S) = \text{Rng}(R) \cap \text{Rng}(S)$
22. $\text{Rng}(R - S) = \text{Rng}(R) - \text{Rng}(S)$
23. $\text{Rng}(R \cup S) = \text{Rng}(R) \cup \text{Rng}(S)$

Problems 24-31: Properties of Binary Relations

Determine whether the following relations are reflexive, symmetric, or transitive. Prove your claims.

- *24. $D = \{(x, x) : x \in S\}$, the diagonal of $S \times S$, where S is any set.
25. \sim , where $P \sim Q$ iff $P \wedge Q$ is a tautology, P and Q being sentences of *Sentential Logic*.
- *26. U , where $x U y$ iff $x^2 + y^2 = 1$, x and y being real numbers.
27. C , where $x C y$ iff $|x - y| < 1$, x and y being real numbers.
28. $<$, where $x < y \leftrightarrow x \in y$, x and y being natural numbers, defined as in Exercise 5.3-58–5.3-61.
29. R , where $x R y \leftrightarrow x \subseteq y$, x and y belonging to the power set $\mathcal{P}(S)$ of some set S .
30. R , where $x R y$ for planar figures x and y iff x and y have the same area.
31. \sim , where $x \sim y$ for straight lines x and y iff x and y are distinct and parallel.

Problems 32-48: Basic Properties and Set Theory

Prove or disprove the following results regarding the interaction of properties of binary relations with the operations of Set Theory.

32. If R is a reflexive relation on S , then so is any subset of R .
- *33. If R is a symmetric relation on S , then so is any subset of R .
34. If R is a transitive relation on S , then so is any subset of R .
35. If \sim is an equivalence relation on S and $T \subseteq S$, then \sim restricted to T is an equivalence relation on T .
36. If \sim is an equivalence relation on S and $T \supseteq S$, then \sim considered as a relation on T is an equivalence relation on T .
- *37. If R is a reflexive relation on S , then so is any superset of R inside $S \times S$.
38. If R is a symmetric relation on S , then so is any superset of R inside $S \times S$.
39. If R is a transitive relation on S , then so is any superset of R inside $S \times S$.

40. If R_1 and R_2 are reflexive relations on S_1 and S_2 respectively, then $R_1 \cap R_2$ is a reflexive relation on $S_1 \cap S_2$.
41. If R_1 and R_2 are symmetric relations on S_1 and S_2 respectively, then $R_1 \cap R_2$ is a symmetric relation on $S_1 \cap S_2$.
42. If R_1 and R_2 are transitive relations on S_1 and S_2 respectively, then $R_1 \cap R_2$ is a transitive relation on $S_1 \cap S_2$.
43. If R_1 and R_2 are reflexive relations on S_1 and S_2 respectively, then $R_1 - R_2$ is a reflexive relation on $S_1 - S_2$.
44. If R_1 and R_2 are symmetric relations on S_1 and S_2 respectively, then $R_1 - R_2$ is a symmetric relation on $S_1 - S_2$.
45. If R_1 and R_2 are transitive relations on S_1 and S_2 respectively, then $R_1 - R_2$ is a transitive relation on $S_1 - S_2$.
46. If R_1 and R_2 are reflexive relations on S_1 and S_2 respectively, then $R_1 \cup R_2$ is a reflexive relation on $S_1 \cup S_2$.
47. If R_1 and R_2 are symmetric relations on S_1 and S_2 respectively, then $R_1 \cup R_2$ is a symmetric relation on $S_1 \cup S_2$.
48. If R_1 and R_2 are transitive relations on S_1 and S_2 respectively, then $R_1 \cup R_2$ is a transitive relation on $S_1 \cup S_2$.

Problems 49-52: Congruence Modulo n

The following problems pertain to the relationship of congruence mod n , defined on \mathbb{Z} as follows:

DEFINITION: Let a and b be integers and let n be a positive integer. Then $a \equiv_n b$ iff $n \mid (a - b)$.

49. Show that $2 \mid (x - y)$ iff x and y have the same parity; i.e., either both x and y are even or both are odd.
- *50. Show that \equiv_n is an equivalence relation.
51. *Alternative Characterization of Congruence Mod n*
Show that $a \equiv_n b$ iff a and b both have the same non-negative remainder when divided by n .
- EC 52. Let \mathbb{Z}_n denote the equivalence classes $[0], [1], \dots, [n - 1]$ for the equivalence relation \equiv_n . Define an operation \oplus of addition on these classes by $[a] \oplus [b] = [a + b]$.
 - a. Show that this operation is well-defined; that is, show that the result is independent of the representative chosen: if $[a] = [a']$ and $[b] = [b']$, then $[a] \oplus [b] = [a'] \oplus [b']$.
 - b. Tell why \oplus is commutative and associative.
 - c. What is the identity element for \oplus ? What is the additive inverse for any element $[a]$?

Problems 53-61: Independence of Basic Relation Properties

Exhibit relations on sets of your choice having the following properties and show that they have them. You may either use familiar relations or construct them to suit the task (specific relations on $S = \{0, 1, 2\}$ will suffice).

53. Not reflexive, not symmetric, and not transitive.
54. Not reflexive, not symmetric, and transitive.
55. Not reflexive, symmetric, and not transitive.
- *56. Not reflexive, symmetric, and transitive.
57. Reflexive, not symmetric, and not transitive.
- *58. Reflexive, not symmetric, and transitive.
- *59. Reflexive, symmetric, and not transitive.
60. Reflexive, symmetric, and transitive.

- EC 61. Criticize the following proof, which purports to show that every symmetric, transitive relation is reflexive. Suppose R is both symmetric and transitive, and let x and y be elements of S such that $x R y$. Then by the symmetric property, $y R x$ too. By the transitive property, then, $x R x$. Thus R is reflexive. ■

Problems 62-67: Properties of Converse Relations

Symbolically formulate and prove the following results pertaining to converse relations (Definition 7).

62. The converse of the converse relation is the original relation.
 63. The converse of the intersection of two relations is the intersection of their converses.
 64. The converse of the difference of two relations is the difference of their converses.
 65. The converse of the union of two relations is the union of their converses.
 66. A relation is symmetric iff it is identical with its converse.
 67. The converse of an equivalence relation is an equivalence relation.

Problems 68-79: Equivalence Classes and Partitions

Describe and picture in some way, if possible, the partitions induced by the following equivalence relations.

- *68. $m \sim n \leftrightarrow 7 \mid (m - n)$, where m and n are natural numbers.
 *69. $(x_1, y_1) \sim (x_2, y_2) \leftrightarrow x_1^2 + y_1^2 = x_2^2 + y_2^2$, where all values are real numbers.
 70. $x \sim y$ iff $\lfloor x \rfloor = \lfloor y \rfloor$.
 71. $l \sim m$ iff l and m are parallel (or identical) straight lines.
 72. $T_1 \sim T_2$ iff T_1 and T_2 are congruent triangles.
 73. $p \sim q$ iff p and q are polynomials of the same degree.
 74. $f \sim g$ iff f and g are polynomials in which $f = c \cdot g$ for some non-zero real number c .
 75. $f \sim g$ iff f and g are functions having the same derivative.
 76. $C \sim D$ iff C and D are playing cards having the same face-value.
 77. $U \sim V$ iff U and V are Rook cards having the same point-value contribution toward the score.
 78. $P \sim Q$ iff P and Q are citizens of the United States having the same birthday and birth year.
 79. *Theorem 1*
 a. Fill the gap in the proof of *Theorem 1a*, by showing that $z \in [y] \rightarrow z \in [x]$. Refer back to the proof of $z \in [x] \rightarrow z \in [y]$ only if you run stuck.
 b. For *Theorem 1b*, show that the common partition cell T to which two related elements x and y belong must be unique.
 c. Show that the relation $\sim_{\mathcal{P}}$, defined in the proof of *Theorem 1b*, is an equivalence relation. Point out where both parts of the definition of a partition are used in your argument.

Problems 80-81: Composition Results

The following results pertain to composition of relations.

- EC 80. *Transitivity and Composition*
 Prove that a relation R on a set S is transitive iff $R^2 \subseteq R$, where $R^2 = R \circ R$. Give a counterexample to show that R^2 need not equal R for transitive R .
- EC 81. *Decomposition of Functions*
 Let $f: S \rightarrow T$ be any function. Show that f can be “factored” into a composition $f = g \circ p$ where p is surjective (onto) and g is injective (one-to-one) by working the following parts.
 a. Define $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Show that \sim is an equivalence relation and identify the equivalence classes.
 b. Let S/\sim denote the set of equivalence classes of \sim in S . Prove that the association p mapping each element x in S to its equivalence class $[x]$ in S/\sim is a surjective function.

- c. Prove that the association g which maps each equivalence class $[x]$ in S/\sim to $f(x)$ is an injective function.
- d. Prove that $f = g \circ p$.

Problems 82-83: Number Systems and Classical Equivalence Relations

The following problems explore some classical equivalence relations in the construction of number systems from simpler ones.

EC 82. *Constructing the Integers*

The relation \sim is defined on $\mathbb{N} \times \mathbb{N}$ as follows: $(a, b) \sim (c, d)$ iff $a + d = b + c$. [Thus, $(a, b) \sim (c, d)$ iff the ordered pairs have the same integer differences: $a - b = c - d$ as integers.]

- a. Show that \sim is an equivalence relation on $\mathbb{N} \times \mathbb{N}$ by proving from its definition that it is reflexive, symmetric, and transitive.
- b. How would you characterize the partition/equivalence classes induced by this relation? How can these equivalence classes be used to model the integers? Give a uniform way to match integers via a one-to-one correspondence to these equivalence classes. (This is the classical way in which the integers are set-theoretically constructed from the natural numbers.)

83. *Constructing the Rationals*

The relation \sim is defined on $\mathbb{Z} \times \mathbb{Z}^*$ as follows: $(a, b) \sim (c, d)$ iff $a \cdot d = b \cdot c$, where a and c are integers and b and d are non-zero integers. [Thus, $(a, b) \sim (c, d)$ iff the ordered pairs have the same quotient $a/b = c/d$ as rational numbers.]

- a. Show that \sim is an equivalence relation on $\mathbb{Z} \times \mathbb{Z}^*$ by proving from its definition that it is reflexive, symmetric, and transitive.
- b. How would you characterize the partition/equivalence classes induced by this relation? How can these equivalence classes be used to model the rational numbers? Give a uniform way to match rational numbers via a one-to-one correspondence to these equivalence classes. (This is the classical way in which the rational numbers are set-theoretically constructed from the integers.)

Problems 84-90: Transitive Extension and Closure

Let R be a binary relation on a set S .

Definition: The *transitive extension* of R is $\overline{R} = R \cup \{(x, z) : (x, y) \in R \text{ and } (y, z) \in R \text{ for some } y \in S\}$.

Definition: The *transitive closure* of R is $R^* = \bigcup_{i=0}^{\infty} R_i$, where $R_0 = R$ and $R_{n+1} = \overline{R}_n$.

- 84. If $S = \{a, b, c, d\}$ and $R = \{(a, b), (b, c), (c, b), (d, a)\}$, determine \overline{R} and R^* .
- EC 85. Is the transitive extension of a binary relation transitive? Prove it in general, or give a counterexample.
- EC 86. Is the transitive closure of a binary relation transitive? Prove it in general, or give a counterexample.
- 87. Prove that $R^* \subseteq T$, where T is any transitive relation containing R .
- 88. Prove that the intersection of all transitive relations containing R is a transitive relation containing R . Conclude from this and Problem 86 that R^* is the intersection of all transitive relations containing R .
- 89. Prove that the transitive closure of a symmetric relation is symmetric.
- 90. Will it ever be necessary to take infinitely many unions to get R^* , or will finitely many suffice (i.e., eventually $R_{n+1} = R_n$). Prove that finitely many are enough, or find a counterexample to show that infinitely many are needed.
- EC 91. Develop an algorithm for constructing the smallest equivalence relation containing a given relation R on a set S .

HINTS TO STARRED EXERCISES 6.3

1. Review the definition of a binary relation.
2. Review the definition of a binary relation.
6. Review the definition of a binary relation.
9. a. Remember that each relation on S is associated with a subset of $S \times S$. How many subsets of $S \times S$ are there?
b. A reflexive binary relation on S needs to contain all ordered pairs of the form (x, x) for $x \in S$. Besides this core of n ordered pairs, it may contain any number of other ordered pairs from $S \times S$; each different subset of these joined to the core will give a distinct reflexive relation. Count how many of these there are.
10. [No hint.]
24. The proofs here follow very quickly from the definitions of reflexive, symmetric, and transitive relations.
26. The proof here follows very quickly from the appropriate definition.
33. No fancy sets or relations are needed to concoct a counterexample here.
37. This result follows quickly using the set theoretic definition of a relation and the appropriate definitions.
50. Review the definition of equivalence relation.
56. A relation containing only one properly chosen ordered pair will do for this one.
58. A very familiar numerical relation has these properties.
59. If you let $S = \{0, 1, 2\}$, removing some well chosen pairs from $S \times S$ will provide a counterexample.
68. Exercises 49 and 50 may be of some help here.
69. Don't get frustrated if you find yourself going around in circles on this one. (Sorry 'bout that!)
80. You can prove both directions of this biconditional by unraveling all the relevant definitions.
82. This problem is EC, but math majors ought to try it sometime in their educational career. It's a classic construction of \mathbb{Z} from \mathbb{N} via set theory. Even more can be done by defining arithmetic operations on the resulting integer quotient structure.