1-2016

Discrete Mathematics: Chapter 4, Basic Set Theory & Combinatorics

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Abstract
The next two chapters deal with Set Theory and some related topics from Discrete Math-ematics. This chapter develops the basic theory of sets and then explores its connection with combinatorics (adding and multiplying; counting permutations and combinations), while Chapter 5 treats the basic notions of numerosity or cardinality for finite and infinite sets.

Most mathematicians today accept Set Theory as an adequate theoretical foundation for all of mathematics, even as the gold standard for foundations.* We will not delve very deeply into this aspect of Set Theory or evaluate the validity of the claim, though we will make a few observations on it as we proceed. Toward the end of our treatment, we will focus on how and why Set Theory has been axiomatized.

But even disregarding the foundational significance of Set Theory, its ideas and terminology have become indispensable for a large number of branches of mathematics as well as other disciplines, including parts of computer science. This alone makes it worth exploring in an introductory study of Discrete Mathematics.

Keywords
set theory, combinatorial probabilities, addition, multiplication, counting

Disciplines
Christianity | Computer Sciences | Mathematics

Comments
- From Discrete Mathematics: An Integrated Approach, a self-published textbook for use in Math 212

✠ Chapter 4 ✠

BASIC SET THEORY
&
COMBINATORICS

4.1 Basic Relations and Operations on Sets

The next two chapters deal with *Set Theory* and some related topics from *Discrete Mathematics.* This chapter develops the basic theory of sets and then explores its connection with combinatorics (adding and multiplying; counting permutations and combinations), while Chapter 5 treats the basic notions of numerosity or cardinality for finite and infinite sets.

Most mathematicians today accept *Set Theory* as an adequate theoretical foundation for all of mathematics, even as the gold standard for foundations.* We will not delve very deeply into this aspect of *Set Theory* or evaluate the validity of the claim, though we will make a few observations on it as we proceed. Toward the end of our treatment, we will focus on how and why *Set Theory* has been axiomatized.

But even disregarding the foundational significance of *Set Theory*, its ideas and terminology have become indispensable for a large number of branches of mathematics as well as other disciplines, including parts of computer science. This alone makes it worth exploring in an introductory study of *Discrete Mathematics.*

Historical Background to Set Theory

Sets arise both from below, as it were, through aggregation (collecting individual things together into a unified whole), and from above, via classification (forming a class via some defining property). Sets occur in everyday life (a set of dishes; a collection MP3 music videos) as well as in science and mathematics (the class of all mammals; the set of all prime numbers). Notwithstanding the usefulness of sets, a *theory* about sets is a relatively recent phenomenon, being about a century and a half old. *Set Theory* did *not* have its origin in ancient civilizations, even if nomadic shepherds kept track of herds by making one-to-one correspondences between their animals and collections of pebbles stored in a pouch. Classifying and collecting were indeed important activities, but treating categories or collections as sets to be operated on, as conceptual objects in their own right, played no real part in the development of mathematics until the late nineteenth century. It was only then that a genuinely useful role was discovered for sets.

The British mathematicians George Boole and Augustus De Morgan made some use of sets mid-century in connection with their treatment of logic, but it was only with the work of the German mathematicians Richard Dedekind and especially Georg Cantor in the last quarter of the century that the value of sets was truly recognized. Dedekind used sets to develop the real number foundations of calculus and also to characterize the natural number system. Cantor was led to develop his theory of infinite sets by research into non-convergent point sets for various Fourier series; using *Set Theory* he was able to settle some open questions in this part of analysis. Over the years he developed *Set Theory* into a branch of mathematics, the centerpiece being his treatment of transfinite (infinite) sets.

Set Theory was considered valuable for mathematics not only because of its analysis of infinity. Some (though not all) mathematicians also touted it for its ability to lay a unified theoretical foundation for all of mathematics. Developments along this line were encouraged by certain schools of thought in the early twentieth century (by logicism, led by Bertrand Russell; and by formalism, led by David Hilbert) and were programmatically developed a bit later by a group of prominent mathematicians writing under the French pseudonym Nicolas Bourbaki.

New Math proponents jumped on the bandwagon in the 1960s. They believed school mathematics could be learned more quickly and economically (crucial concerns to the U.S. at that time in its space race with the U.S.S.R.) if children were exposed to the conceptual

* However, there are also advocates for another more algebraic foundation, and some would be content with simply having different foundations for different parts of mathematics.

structure of mathematics. Drawing from developments earlier in the century, they promoted *Set Theory* as the theoretical basis and unifying apparatus for mathematics. In the following years, though, a strong reaction set in. Many educators resisted teaching something as abstract as *Set Theory* to young children. Consequently, sets now play only a minimal role in most elementary mathematics programs; where they do come in, they are now taught in moderation and far more concretely than earlier.

Regardless of the validity of using *Set Theory* as the ultimate foundation for mathematics or as a unifying tool for all of mathematics, *Set Theory* has played a very important role in contemporary mathematics, including *Discrete Mathematics*. We will take it up here as background for a number of topics treated later in the book.

The Idea, Notation, and Representation of Sets

A *set* is any definite collection of things of any kind whatsoever. This isn't really a definition; it merely uses the synonym *collection* to say what a set is. In fact, no genuine definition can be given. The term *set* refers to something so basic it needs to be taken as primitive, as an undefined term. Everyone is familiar with its concrete meaning from everyday experience; *Set Theory* sharpens and extends this intuition. Other synonyms for 'set' are 'family', 'class', 'group', 'herd', and so on. Certain terms will be more appropriate than others in given situations – you wouldn't call a sports team a gaggle of players – but the basic idea is always the same: anytime a multiplicity of distinct and definite objects are gathered together into a single conceptual unit, you have a set. How the *elements* or *members* of a set are related to one another is irrelevant from a simple set-theoretical point of view. Whatever algebraic and relational structures a set might have are irrelevant on this level; basic *Set Theory* is only concerned with which things belong to what sets.

In formulating assertions about sets, it is customary to use capital letters to indicate sets and lower case letters to stand for individual members of such sets, though there are times when this practice must be abandoned. The symbol $\in$ is used to indicate set membership. Thus, if P denotes the set of all prime numbers, $3 \in P$ says that 3 is a prime number; $x \in \mathbb{Q}$ says that x belongs to the set of rational numbers $\mathbb{Q}$.

A set may be specified in two main ways: by listing its elements (giving a membership roster) or by stipulating a membership criterion (a common property shared by all and only those elements in the set). This leads to two different ways to denote specific sets. If the sets are small enough, their members can be listed between braces, separating the different elements with commas. The set of primes less than ten can be denoted by $\{2, 3, 5, 7\}$. Sometimes dots are used to list the elements of a set, provided the pattern generated is clear from the elements that are present. Thus, the first one hundred counting numbers can be written as $\{1, 2, 3, \cdots, 99, 100\}$ and the entire set of natural numbers as $\{0, 1, 2, 3, \cdots\}$.

Denoting a finite set by *Set Roster Notation* can be taken as a shorthand for a disjunction explicitly identifying all of its members. Thus $S = \{a_1, a_2, \ldots, a_n\}$ means that $x \in S$ iff $x = a_1 \vee x = a_2 \vee \cdots \vee x = a_n$.

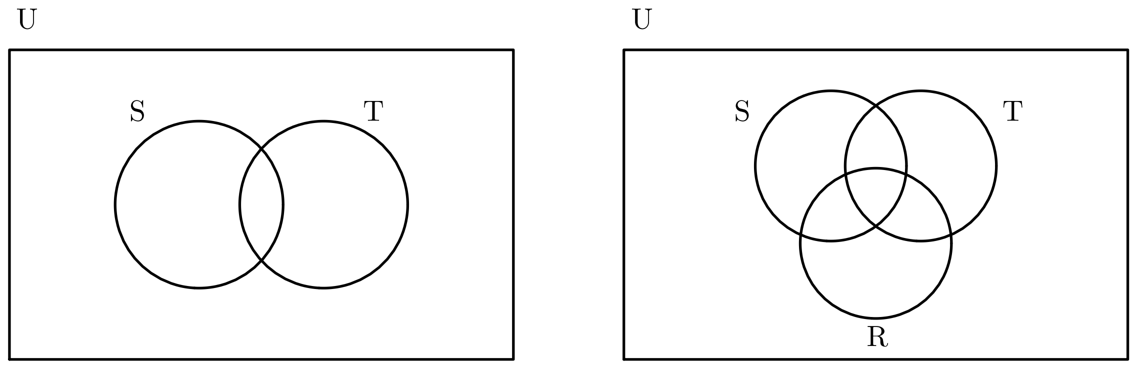
Sets that are too large to list conveniently or that cannot be listed at all may be indicated by means of *Set Descriptor Notation*. Braces are again used, but now a variable representing an arbitrary element of the set is given together with a description or formula that all set members satisfy. For example, the notation $\{x : x \text{ is prime}\}$ identifies the set of all prime numbers. The notation $\{m/n : m, n \text{ integers}, n \neq 0\}$ indicates the set of all rational numbers. The abstract assertion $S = \{x : P(x)\}$ can be taken as equivalent to the proposition $\forall x(x \in S \leftrightarrow P(x))$.*

Often *Restricted Set Descriptor Notation* is used to present a set. $S = \{x \in U : P(x)\}$ indicates that S consists of all those elements inside U satisfying statement $P(x)$. Here U

* More formally, we would state this as a definition of our notation: $S = \{x : P(x)\} \leftrightarrow \forall x(x \in S \leftrightarrow P(x))$.

functions as a restricting universe of discourse.* Thus $S = \{x \in U : P(x)\}$ is an abbreviated equivalent for the membership claim $\forall x(x \in S \leftrightarrow x \in U \land P(x))$.

To assist us as we state and prove theorems of *Set Theory*, we will use diagrams to represent sets pictorially. These diagrams play the same role in *Set Theory* that geometric diagrams do in geometry: they help us follow what is being asserted, but they are not a substitute for deductive argumentation. Such diagrams are called *Venn Diagrams* after the late nineteenth century English logician John Venn, who introduced them into logic, though similar devices were used earlier by other mathematicians.**



Venn Diagrams

Venn diagrams typically contain two or three circles located within a single rectangle. The outer rectangle represents the universe of discourse being considered, and the circles inside stand for particular sets. When arbitrary sets are intended, the circles are drawn as overlapping to permit all possible relations among the sets (see above). Overlapping regions do not automatically indicate shared membership; the existence of intersecting regions only permits that as a possibility. Particular sets and the existence of elements within a given region may be indicated either by means of shading or listing members.

Equal Sets and Subsets

Since sets are completely determined by their members, it is intuitively obvious that two sets should be considered equal iff they contain exactly the same elements. This gives us the following axiom or definition.†

Axiom/Definition 4.1-1: Equality for Sets
$S = T \leftrightarrow \forall x(x \in S \leftrightarrow x \in T)$

Equality between two sets S and T is thus demonstrated, according to this definition, by taking an arbitrary element x (to satisfy *UG*) and proving the biconditional $x \in S \leftrightarrow x \in T$. This in turn is usually done (via *BI*) by two subproofs: supposing $x \in S$, you prove $x \in T$;

* In Section 5.3 we will see that there may be good reasons for restricting sets to those that can be formed inside other already existing sets.

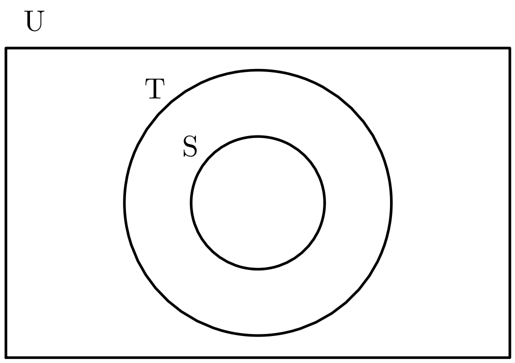
** Tracing this usage back, Venn attributes them to Euler. Euler probably got them from his teacher, Jean Bernoulli, who in turn was likely indebted to his collaborator Leibniz. Leibniz used them in his work to exhibit relations among classes, just as we do. Earlier versions prior to Venn were somewhat more limited than what we currently use.

† There is more here than meets the eye, since we already have a fixed, logical interpretation for "equals." The forward part of this definition follows from the inference rules for identity and so can actually be proved; the backwards part, on the other hand, needs to be asserted as an axiom. Since taken together it specifies how we will be using equality in the context of *Set Theory*, we will treat it here as a definition of equality for sets.

then, supposing $x \in T$, you prove $x \in S$. At times, however, you may be able to chain a number of biconditionals together and so establish both directions simultaneously.

Arguing for the two component conditionals $x \in S \rightarrow x \in T$ and $x \in T \rightarrow x \in S$ in the subproofs for *BI* amounts to showing in each case that the first set mentioned is contained in the second one as a subset. This leads us to the next definition and to our first proposition.

DEFINITION* 4.1 - 2: *Subset and Superset Inclusions
(a) *Subset:* $S \subseteq T \leftrightarrow \forall x(x \in S \rightarrow x \in T)$
(b) *Superset:* $T \supseteq S \leftrightarrow S \subseteq T$



$S \subseteq T$

In order to show that a set S is a subset of a set T, therefore, you must show for each x that if it is in S, then it is in T; i.e., letting x represent an arbitrary element of S, you must prove that x is also an element of T. We apply this proof procedure as well as the one for equality in the proof of the first proposition. Proofs for the first few propositions will be talked through and worked in more detail than typical. Try to supply the reasons at the question prompts. Once you see what needs doing in such proofs, you can begin to abbreviate them.

PROPOSITION* 4.1 - 1: *Equality and Subset Inclusion
$S = T \leftrightarrow S \subseteq T \wedge T \subseteq S$

***Proof*:**
Our proposition is a biconditional sentence, so we will use *BI*.

- First suppose $S = T$.
  Then $\forall x(x \in S \leftrightarrow x \in T)$. (why?)
    Suppose that x is any element of S.
    Then x must be an element of T. (why?)
  Hence $S \subseteq T$. (why?)
  Similarly $T \subseteq S$.
  And so $S \subseteq T \wedge T \subseteq S$. (why?)
- Conversely, suppose $S \subseteq T \wedge T \subseteq S$.
  Then $\forall x(x \in S \rightarrow x \in T)$ and $\forall x(x \in T \rightarrow x \in S)$. (why?)
  But then given any element x, $(x \in S \rightarrow x \in T) \wedge (x \in T \rightarrow x \in S)$. (why?)
  This means $\forall x(x \in S \leftrightarrow x \in T)$. (why?)
  And so $S = T$. (why?)

This proves the proposition. (why?) ■

The main way in which sets are shown to be equal is via the definition, using arbitrary elements of the sets to establish the equality. However, sometimes it will be possible to remain up on the set level, without descending to the level of the sets' elements. Then *Proposition* 4.1-1 may come in handy. Occasionally, it will also work to show the equality of two sets by taking the set on one side of the equation and showing it via some transformation to be the same as the set on the other side.

The subset relationship is sometimes confused with set membership. This is due to fuzzy thinking. Elements are not subsets. You should take care to keep these two concepts distinct. The number 2 is an element of the set P of prime numbers; it is not a subset of P. On the other hand, the set P of prime numbers is a subset of the natural number system $\mathbb{N}$; it is not an element of $\mathbb{N}$, because it is not a natural number. The potential for confusion on this score is increased when we come to consider sets whose elements are themselves sets (see Section 4.2), but it can be avoided if you stay on your guard.

According to *Proposition* 1, whenever $S \subseteq T$ and $T \subseteq S$, then $S = T$. In technical terms this is summarized by saying that $\subseteq$ is an *antisymmetric* relation ($\leq$ is another such relation). And, like $\leq$ for numbers, the $\subseteq$ relation is not symmetric; that is, it is generally not true that whenever $S \subseteq T$, then $T \subseteq S$. The subset relation does have two other basic properties, however. The first one is completely trivial; the second is less so but should be fairly obvious.

PROPOSITION* 4.1 - 2: *Reflexive Law for Inclusion

$S \subseteq S$

***Proof*:**

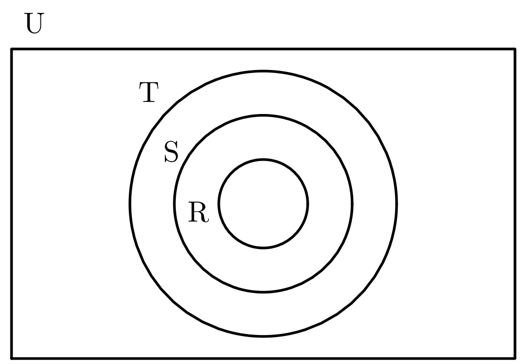
If $x \in S$, then $x \in S$. ■

PROPOSITION* 4.1 - 3: *Transitive Law for Inclusion

$R \subseteq S \wedge S \subseteq T \rightarrow R \subseteq T$

***Proof*:**

This sentence is a conditional sentence, so we'll use *CP* to prove it.
Suppose $R \subseteq S \wedge S \subseteq T$.
Using the *Method of Backward Proof Analysis* [doing this is *crucial*–otherwise you may get lost in the "givens" and not get off on the right foot], note that we *want to show* that $R \subseteq T$. This is done by proving that if $x \in R$, then $x \in T$, too.
Using *CP* as our proof strategy for this, we now suppose $x \in R$.
Since $R \subseteq S$, $x \in S$. (why?)
But $S \subseteq T$, too; so $x \in T$. (why?)
This is what we needed to show, so $R \subseteq T$. ■



$R \subseteq S \subseteq T$

Our definition of subset inclusion permits the possibility that the two sets are equal. There is also a more restricted notion of inclusion. *Proper inclusion* occurs when the subset is strictly smaller than the superset. This relation is also transitive, but it is not reflexive or symmetric (see Exercises 10 − 11).

DEFINITION* 4.1 - 3: *Proper Subset Inclusion

$S \subset T \leftrightarrow S \subseteq T \wedge S \neq T$

Among all possible sets, one set is contained in every set. This is the *empty set*, denoted by $\emptyset$. The set $\emptyset$ plays somewhat the same role in *Set Theory* that the number 0 plays in arithmetic (see Exercises 13 − 16). And, like the number 0, it sometimes gives conceptual trouble when it is first encountered (how can something be a set if it doesn't contain any elements?). It might help you to think of sets as collectors; an empty set is a collector with no objects inside it. We can define it by using a property that cannot be satisfied (a contradictory property).

DEFINITION* 4.1 - 4: *Empty Set

$\emptyset = \{x : x \neq x\}$

Given the meaning of *Set Descriptor Notation*, it immediately follows from this definition that $\forall x(x \notin \emptyset)$: that is, nothing is in $\emptyset$.

You may understand just why each statement was made in the proofs of the first few propositions, since the distance between successive steps was quite small. But don't expect this always to be the case. You should get in the habit of reading a proof with a pencil and paper. The main steps may be there, but you will often have to fill in some details. Knowing what proof strategies are available for the sentences involved should give you insight into what might be going on in a proof and thus make you better able to follow it, even when it is sketchy. The proof of *Proposition* 4 will give you some practice at this (see Exercise 21).

PROPOSITION* 4.1 - 4: *Empty Set Inclusion

$\emptyset \subseteq S$

***Proof*:**

Suppose that $x \notin S$.
But $x \notin \emptyset$, too.
Thus $\emptyset \subseteq S$. ■

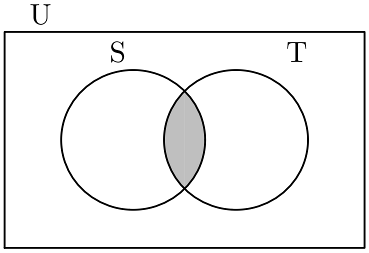
Intersection and Union

The two most basic binary operations on sets are *intersection* and *union*. We will state definitions for these using *Set Descriptor Notation*.

DEFINITION* 4.1 - 5: *Intersection

$S \cap T = \{x : x \in S \wedge x \in T\}$

U



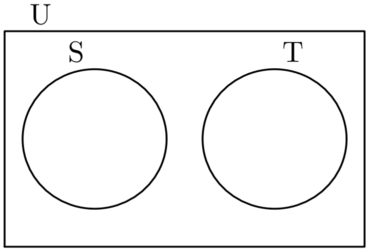
$S \cap T$

We are assuming that intersection is a genuine binary operation on sets: given any two sets S and T, the intersection $S \cap T$ is a well-defined set. This is so even when the two sets have no overlap or are *disjoint*. In this case the intersection is the empty set.

DEFINITION* 4.1 - 6: *Disjoint Sets

S and T are disjoint sets iff $S \cap T = \emptyset$.

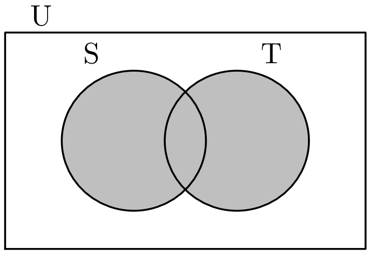
U



$S \cap T = \emptyset$

DEFINITION* 4.1 - 7: *Union

$S \cup T = \{x : x \in S \vee x \in T\}$



$S \cup T$

It is clear from the definitions just given that intersection and union parallel the logical operations of conjunction and disjunction. This correspondence reveals itself more fully in various laws governing these operations, which are the counterparts of various *Replacement Rules* for SL. These propositions are most easily proved by employing this correspondence; if the associated *Replacement Rules* from logic are not used, the proofs can be tedious and rather lengthy. You should also draw Venn diagrams for the sets being equated in each proposition. This does not constitute a genuine proof, but it will help convince you that the propositions are true.

PROPOSITION* 4.1 - 5: *Idempotence Laws for Intersection and Union

a) $S \cap S = S$
b) $S \cup S = S$

***Proof*:**

a) Let x be an arbitrary element.
By *Definition* 5, $x \in S \cap S \leftrightarrow x \in S \wedge x \in S$.
But by *Idem*, $x \in S \wedge x \in S \models\!\!\!\dashv x \in S$.
Substituting, $x \in S \cap S \leftrightarrow x \in S$.
But this means that $S \cap S = S$ by *Definition* 1.

b) Note that part b differs from part a only in the operation involved. Replacing $\cap$ by $\cup$ and $\wedge$ by $\vee$ everywhere in part a's argument, a proof for b immediately results. ■

The way in which the second half of the last proposition was proved suggests that a *Duality Principle* may be at work in *Set Theory*: replace $\cap$ with $\cup$ and conversely and you have a new proposition and a new proof technique. The following propositions seem to offer further confirmation of this possibility, but does such a principle really hold? We'll return to answer this question later (see also Exercise 4.2-38).

PROPOSITION* 4.1 - 6: *Commutative Laws for Intersection and Union

a) $S \cap T = T \cap S$
b) $S \cup T = T \cup S$

***Proof*:**

a) $x \in S \cap T \leftrightarrow x \in S \wedge x \in T$ [Defn 5]
$\leftrightarrow x \in T \wedge x \in S$ [*Comm* ($\wedge$)]
$\leftrightarrow x \in T \cap S$. [Defn 5]

b) See Exercise 18a. ■

PROPOSITION* 4.1 - 7: *Associative Laws for Intersection and Union
a) $R \cap (S \cap T) = (R \cap S) \cap T$
b) $R \cup (S \cup T) = (R \cup S) \cup T$

***Proof*:**
See Exercises 17a and 18b. ■

PROPOSITION* 4.1 - 8: *Distributive Laws: Intersection/Union over Union/Intersection
a) $R \cap (S \cup T) = (R \cap S) \cup (R \cap T)$
b) $R \cup (S \cap T) = (R \cup S) \cap (R \cup T)$

***Proof*:**
See Exercise 19. ■

PROPOSITION* 4.1 - 9: *Absorption Laws and Subset Ordering
a) $S \cap T \subseteq S \qquad S \cap T \subseteq T$
b) $R \subseteq S$ and $R \subseteq T$ iff $R \subseteq S \cap T$.
c) $S \subseteq S \cup T \qquad T \subseteq S \cup T$
d) $R \subseteq T$ and $S \subseteq T$ iff $R \cup S \subseteq T$.

***Proof*:**
We will sketch proofs for the first two parts and leave the others for the exercises (see Exercise 18cd).

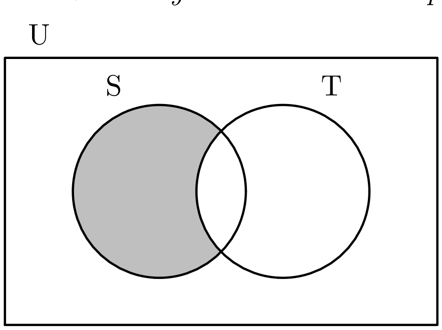
a) This is essentially the set theoretic counterpart of the SL inference rule *Simp*: $x \in S \wedge x \in T \vDash x \in S, \quad x \in S \wedge x \in T \vDash x \in T$.
b) R is contained in both S and T iff all its elements are in both S and T. This occurs iff its elements are contained in $S \cap T$; i.e, iff $R \subseteq S \cap T$. ■

The first two parts of the last proposition can be summarized by saying that the intersection $S \cap T$ is the largest set contained in both S and T; similarly, parts *c* and *d* can be summarized by saying that the union $S \cup T$ is the smallest set containing both S and T. These results are both important for the theory of ordering sets according to the subset relation. We will explore these and related matters in a more algebraic setting in Chapter 7.

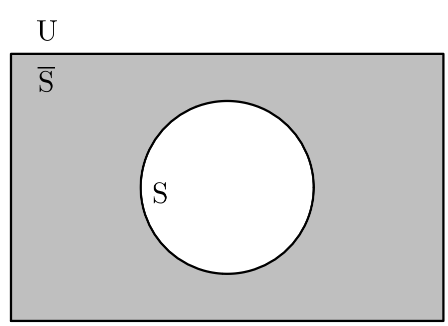
Set Difference and Set Complement

Given two sets S and T, we can not only form their union and intersection; we can also take their set difference. And given a set U, we can define set complement relative to U in terms of set difference. These notions are defined as follows, using *Set Descriptor Notation*.

DEFINITION* 4.1 - 8: *Set Difference, Set Complement
a) $S - T = \{x : x \in S \wedge x \notin T\}$.
b) *Let U be any set. Then the complement of S inside U is $\overline{S} = U - S$.*



$S - T$



$\overline{S}$

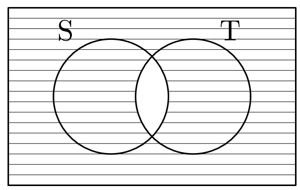
The notion of set complement does not require $S \subseteq U$, but this is the context in which it is usually applied. U thus forms a sort of universal superset for sets whose complements are being formed. The following proposition, however, holds in all cases, regardless of how S, T, and U are related. Note once again how the counterpart of a Replacement Rule plays the crucial role. Other results involving set difference and set complement are found in the exercises.

PROPOSITION* 4.1 - 10: *De Morgan's Laws for Set Complement

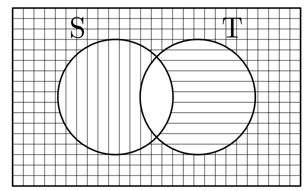
Let S and T be any two sets, with their complements being taken with respect to a common set U. Then

a) $\overline{S \cap T} = \overline{S} \cup \overline{T}$;

b) $\overline{S \cup T} = \overline{S} \cap \overline{T}$.



$\overline{S \cap T}$



$\overline{S} \cup \overline{T}$

***Proof*:**

a) The above diagrams make it seem plausible that the complement of the intersection is the union of the complements.
The argument for this claim goes as follows:

$x \in \overline{S \cap T} \leftrightarrow x \in U \wedge x \notin (S \cap T)$	[Defn of set complement]
$\leftrightarrow x \in U \wedge \neg(x \in S \wedge x \in T)$	[Defn of intersection]
$\leftrightarrow x \in U \wedge (x \notin S \vee x \notin T)$	[DeM for negated $\wedge$]
$\leftrightarrow (x \in U \wedge x \notin S) \vee (x \in U \wedge x \notin T)$	[Distributive Law: $\wedge$ over $\vee$]
$\leftrightarrow x \in U - S \vee x \in U - T$	[Defn of set difference]
$\leftrightarrow x \in (U - S) \cup (U - T)$	[Defn of union]
$\leftrightarrow x \in \overline{S} \cup \overline{T}$.	[Defn of set complement]
Therefore $\overline{S \cap T} = \overline{S} \cup \overline{T}$. ■	[Defn of set equality]

b) See Exercise 20a.

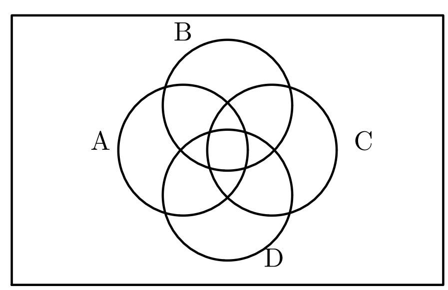
EXERCISE SET 4.1

To work the following problems, you do ***not*** *need to use a two-column format nor only logical rules of inference as reasons. Use the deduction systems of PL and SL as* ***guides*** *to help you determine proof strategy. Illustrate your results using Venn diagrams wherever appropriate.*

*1. *Venn Diagrams*

*a. A certain company presented a set of data having four possible categories A, B, C, and D by means of the Venn diagram at the right. Explain why this diagram is deficient.

EC b. Modify the diagram in some way to show all possible regions.



Problems 2 - 5: Illustrating Basic Operations

Let $U = \{x \in \mathbb{N} : x \leq 30\}$, $E = \{x \in U : x \textit{ is even}\}$, $O = \{x \in U : x \textit{ is odd}\}$, *and* $P = \{x \in U : x \textit{ is prime}\}$. *Determine the following sets.*

*2. *Intersections*
a. $O \cap E$ *b. $O \cap P$ c. $E \cap P$

*3. *Unions*
a. $O \cup E$ b. $O \cup P$ *c. $E \cup P$

*4. *Complements Relative to U*
*a. $\overline{E}$ b. $\overline{O}$ c. $\overline{P}$ *d. $\overline{O \cup P}$

*5. *Set Differences*
a. $E - O$ *b. $E - P$ c. $P - E$ d. $O - P$

Problems 6 - 8: True or False

Are the following statements true or false? Explain your answer.

*6. Two sets are equal iff each one has elements of the other.

*7. $\{1, 2, 3\} = \{3, 2, 1\}$

8. The complement of the intersection of two sets is the intersection of the complements.

Problems 9 - 12: Proper Subset Inclusion

Prove the following properties.

9. Proper Containment: $S \subset T \leftrightarrow S \subseteq T \wedge \exists x(x \in T \wedge x \notin S)$

10. Non-Reflexivity: $S \not\subset S$

11. Non-Symmetry: $S \subset T \rightarrow T \not\subset S$

12. Transitivity: $S \subset T \wedge T \subset R \rightarrow S \subset R$

Problems 13 - 16: Properties of the Empty Set

Prove the following.

13. $S \subseteq \emptyset \rightarrow S = \emptyset$

14. $\emptyset \cap S = \emptyset = S \cap \emptyset$

15. $\emptyset \cup S = S = S \cup \emptyset$

16. $S - \emptyset = S; \quad \emptyset - S = \emptyset$

Problems 17 - 21: Proofs

Prove the following propositions concerning set theoretical operations. First construct a Venn diagram to illustrate the proposition, then give an argument for it. Where one exists, the associated Replacement Rule of SL should be of real help.

17. *Intersection*
a. *Proposition* 4.1-7a

18. *Unions*
a. *Proposition* 4.1-6b
b. *Proposition* 4.1-7b
c. *Proposition* 4.1-9c
d. *Proposition* 4.1-9d

*19. *Intersection and Union*
*a. *Proposition* 4.1-8a
b. *Proposition* 4.1-8b

*20. *Complements*
*a. *Proposition* 4.1-10b

21. Analyze and rewrite the proof of *Proposition* 4.1-4, filling in any steps you think are still required to make the proof more understandable. Identify the various proof strategies used in asserting each statement in the proof.

Problems 22 - 54: Theorems or Not?

Determine whether the following claims are theorems of Set Theory or not. Use Venn diagrams to help you decide. If they are true, prove them; if they are not, provide a specific counterexample. If you see an obvious way to fix a false result, restate it and then prove it. Assume that complements are taken relative to some universal set U containing R, S, and T.

*22. $R \cup S = R \cup T \rightarrow S = T$

23. $R \cap S = R \cap T \rightarrow S = T$

*24. $R \subseteq S \rightarrow R \cap T \subseteq S \cap T$

25. $R \subseteq S \rightarrow R \cup T \subseteq S \cup T$

26. $S \subseteq T \leftrightarrow S \cup T = T$

*27. $S \subseteq T \leftrightarrow S \cap T = S$

*28. $R \subseteq T \vee S \subseteq T \leftrightarrow R \cap S \subseteq T$

29. $R \subseteq S \vee R \subseteq T \leftrightarrow R \subseteq S \cup T$

30. $S \cap T = S \leftrightarrow S \cup T = T$

31. $S \cap T = \emptyset \rightarrow S = \emptyset \vee T = \emptyset$

32. $S \cup T = \emptyset \leftrightarrow S = \emptyset \wedge T = \emptyset$

33. $S - T \subseteq S$

34. $S - T = S \cap \overline{T}$

35. $S - T = S - (S \cap T)$

*36. $S \cup T = (S - T) \cup (T - S)$

37. $(S - T) \cap T = \emptyset$

38. $S - T = \overline{T} - \overline{S}$

39. $S \cap T = \emptyset \leftrightarrow S - T = \emptyset$

40. $T - S = T - R \rightarrow S = R$

41. $S - R = T - R \rightarrow S = T$

42. $S \subseteq T \rightarrow S - R \subseteq T - R$

43. $R \subseteq S \rightarrow T - S \subseteq T - R$

EC 44. $R \cup S = R \cup T \leftrightarrow S - R = T - R$

45. $T - (S - R) = (T - S) \cup (T \cap R)$

46. $(T - S) - R = (T - R) - S$

47. $(T - S) - R = (T - S) \cap (T - R)$

48. $(T - S) - R = (T - S) - (R - S)$

49. $S - (S - T) = S \cap T$

50. $S - (S - T) = T \leftrightarrow T \subseteq S$

51. $T - (S \cap R) = (T - S) \cap (T - R)$

52. $T - (S \cup R) = (T - S) \cap (T - R)$

*53. $\overline{(\overline{S})} = S$

54. $(T \cup S) - R = (T - R) \cup (S - R)$

Problems 55 - 69: Symmetric Difference

The ***symmetric difference*** *of two sets S and T is defined by $S \sqcup T = (S - T) \cup (T - S)$. With this definition, prove the following results.*

55. $S \sqcup S = \emptyset$
56. $S \sqcup \emptyset = S$
57. $S \sqcup T = T \sqcup S$
58. $(S \sqcup T) \sqcup R = S \sqcup (T \sqcup R)$
59. $S \sqcup T \subseteq S \cup T$
60. $(S \sqcup T) \cap (S \cap T) = \emptyset$
61. $S \subseteq S \sqcup T \leftrightarrow T \subseteq S \sqcup T \leftrightarrow S \cap T = \emptyset \leftrightarrow S \cup T = S \sqcup T$
62. $S \sqcup T = (S \cup T) - (S \cap T)$
63. $\overline{S \sqcup T} = (S \cap T) \cup (\overline{S \cup T})$
64. $S \sqcup T = (S \cup T) \sqcup (S \cap T)$
65. $S \cup T = (S \sqcup T) \sqcup (S \cap T)$
66. $S \cap T = (S \sqcup T) \sqcup (S \cup T)$
67. $S \cap (T \sqcup R) = (S \cap T) \sqcup (S \cap R)$
68. $R \sqcup S = R \sqcup T \rightarrow S = T$
69. $S = T \leftrightarrow S \sqcup T = \emptyset$

HINTS TO STARRED EXERCISES 4.1

1. a. How many distinct regions should there be for four sets? How many are there here? To see what's missing, consider, for example, the region covered by $A \cap C$.

2. b. $O \cap P$ is the set of all numbers that are elements of both O and P.

3. c. $E \cup P$ is the set of all numbers that are elements of either E or P.

4. a. $\overline{E}$ is the set of all elements in U but not in E.
   d. Do this directly, or make use of *De Morgan's Laws* (Proposition 4.1-10).

5. b. $E - P$ is the set of all numbers that are elements of E but not P.

6. [No hint.]

7. [No hint.]

19. a. Make use of an appropriate related *Replacement Rule* of SL. See the proofs of Propositions 6 and 10 for how to formulate your argument.

20. a. Prove that $x \in \overline{S \cup T} \leftrightarrow x \in \overline{S} \cap \overline{T}$ by using what you know about sentential connectives and the definitions of complement, union, and intersection.

22. Can R just be subtracted out of a union to leave the remaining sets? Work this problem (and the following ones) by doing two opposite things: look for an argument to prove the result, and look for a simple concrete counterexample to disprove the result.

24. A strategy to try here uses the definition of being a subset.

27. How are biconditional statements proved? Keep your logic fresh.

28. Begin each half of the proof by focusing on what you want to prove. Use *Cases* for the forward direction; the backwards direction requires more thought.

37. Draw a picture here to start with. If it seems like the result is false, make the sets concrete and you'll have a counterexample.

44. Draw a Venn diagram here before starting on a proof.

53. What *Replacement Rule* from SL does this remind you of? Use that rule in your argument.

4.2 Collections of Sets and the Power Set

Section 4.1 initiated our treatment of elementary *Set Theory*. There we discussed various relations between sets (set equality, subsets) as well as the role of the empty set, and we introduced the most basic binary operations on sets (intersection, union, set difference, set complement). In addition to defining these notions, we stated and proved a number of fundamental properties for them, using what we knew about *Sentential Logic*.

We now continue our exploration on the next level, looking at *Set Theory* topics that involve collections of sets. The sets we form here have sets as their elements; this accounts for the abstractness some experience on this level. We will also take another look at set operations in this broader context. We conclude our investigation by looking at a particular collection of sets, the power set of a set.

Collecting Sets into Sets

So far we have focused more or less on two levels of set theory: sets and their elements. You may think of these as very different things, objects and collections; but nothing prohibits us from taking the sets themselves as elements for a collection on a still higher level. Sets are legitimate entities and so can be collected to form sets of sets.

This occurs in everyday life as well as mathematics. Baseball players are members of teams, which are members of divisions and leagues. Players are not members of leagues, and teams are not subsets of leagues. Leagues provide a third level of set-theoretic reality in the world of major league baseball. Without the ability to form sets of sets, there could be no division run-offs or world series.

But keeping our nose out of sports and down on the grindstone of mathematics, we can see other needs for such sets. To take a simple example, geometric figures are often viewed as infinite sets of points in a certain configuration. A pair of isosceles triangles, therefore, is a collection of two point sets. If they were merely a joined conglomeration of points instead of a set of sets of points, we couldn't say that there were *two* triangles there; it would be an *infinite* collection of points instead.

We can also take an example from number theory, where mathematicians are often interested in what the remainder is when one integer is divided by another.

✠ **EXAMPLE 4.2 - 1**

Discuss the collection of sets that results when natural numbers are divided by the number 4.

Solution

Our universe of discourse here is $\mathbb{N}$. Any natural number can be divided by 4, leaving a remainder or residue of 0, 1, 2, or 3.

We're thus led to form four residue classes: R_0, R_1, R_2, and R_3. For example, $7 \in R_3$ and $16 \in R_0$ because these numbers leave remainders of 3 and 0 respectively.

These sets can be collected into a collection of four residue classes: $\{R_0, R_1, R_2, R_3\}$. It turns out that these residue classes can treated in much the same way as numbers, yielding what is called *modular arithmetic*. We won't go into this development here (see Section 6.3), but you should be aware of the fact that such collections have very important uses.

Sets of sets occur often in abstract settings. In more advanced mathematics courses, many algebraic structures are constructed as so-called *quotient structures*, which are sets of sets of a special type. Courses in analysis and topology consider other sorts of collections as a foundation for defining some of their central notions. So working with collections of sets is an important skill to learn if you are going on in mathematics. Also, without a way to gather a number of sets into a set, we wouldn't be able to collect the subsets of a given set into a set, something we will discuss shortly.

Given any number of sets, then, it seems we ought to be able to collect them together into a set, just as is done with individual objects. We will assume such families of sets exist as sets in good standing. We can denote such sets, as before, either by listing their members or by formulating a property they must all satisfy.

✠ **EXAMPLE 4.2 - 2**

Let S and T be two distinct sets. Discuss:
a) the set whose sole member is a set S,
b) the set whose elements are the sets S and T, and
c) the set whose elements are the sets S and S.

Solution

a) The *singleton* $\{S\}$ consists of a single element. Clearly $S \in \{S\}$, but $S \neq \{S\}$;* nor is either set a subset of the other one. For if S were, for example, the set of all perfect squares, S would contain infinitely many numbers, while $\{S\}$ would contain only a single entity, the collection of these numbers.

b) The *doubleton* $\{S, T\}$ is formed by pairing up the two sets S and T as elements of another set. In general, elements of S and T will not be elements of the doubleton. It thus differs from both S and T; nor can it be obtained from them by taking an intersection, union, or set-difference. It lies on a higher set-theoretic level than these sets. This is certainly the case, for example, when S is the set of even numbers and T is the set of odd numbers (see Exercise 3).
The doubleton $\{T, S\}$ is identical with the doubleton $\{S, T\}$ since they have exactly the same elements. The order in which a set's elements are listed is irrelevant to simple set equality.

c) The *doubleton* $\{S, S\}$ is identical with the *singleton* $\{S\}$; both contain S as their sole element. Multiplicity is irrelevant to *set identity*. An object either is or is not a member of a set; it cannot be doubly present, even if it is listed twice.**

Total Intersections and Unions

The ordinary operations of intersection and union are binary set-theoretic operators: they operate on pairs of sets. By repeating the process, finite intersections and unions can be performed (see Exercises 13-14). However, given *any* collection of sets, whether finite or not, we would like to be able to perform a total intersection or total union on the sets in this collection, all at once. Such intersections and unions are defined as follows.

DEFINITION* 4.2 - 1: *Total Intersections of Collections

If $\mathcal{C}$ is a non-empty collection of sets, then $\bigcap_{S \in \mathcal{C}} S = \{x : (\forall S \in \mathcal{C})(x \in S)\}$.

DEFINITION* 4.2 - 2: *Total Unions of Collections

If $\mathcal{C}$ is a non-empty collection of sets, then $\bigcup_{S \in \mathcal{C}} S = \{x : (\exists S \in \mathcal{C})(x \in S)\}$.

---

* This assumes S is not a set having itself as its sole element. Such anomalous sets are ruled out in axiomatic presentations of *Set Theory*. We will touch on this briefly in Section 5.3.

** There are mathematical entities, called *multisets* or *bags*, however, in which multiplicity is taken into account.

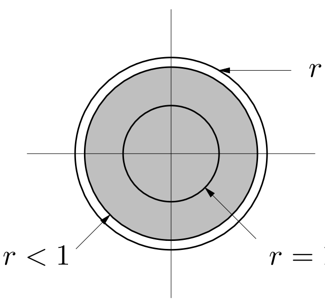
In words, the intersection of a family of sets consists of all the elements that belong to every set in the collection. The union of a collection of sets consists of all those elements that belong to one or more sets in the collection.

✠ **EXAMPLE 4.2 - 3**

Determine the total intersection and total union for the collection $\mathcal{C}$ of concentric closed discs centered about the origin, $D_r = \{(x, y) : x^2 + y^2 \leq r\}$, where $1/2 < r < 1$.

Solution

Since each disc in the collection is centered about the origin, the intersection of any two of them will be the smaller of the two discs. The total intersection would thus be the smallest disc of all, if there were one. However, since the disc radius r is always greater than $1/2$ and there is no smallest number greater than $1/2$, there is no smallest disc in this collection.



The entire disc of radius $1/2$ is certainly contained in all of the discs, but given any disc slightly bigger than this, we can always find a smaller disc in the collection by choosing our radius r a bit closer to $1/2$. So the total intersection of all discs cannot extend beyond $D_{1/2}$, the disc whose radius is $1/2$.

Therefore $\bigcap_{D \in \mathcal{C}} D = D_{1/2}$.

Similarly, given any two discs from the collection, their union is the larger of the two discs. The total union of all the discs would thus be the largest disc, if there were one. There isn't, however, since there is no largest real number less than 1.

The discs can be made arbitrarily close to the unit disc D_1 by choosing r very close to 1, but since $r < 1$ for every disc in the collection, none of the points on the unit circle $x^2 + y^2 = 1$ belongs to any of the collection's discs. Nevertheless, every point strictly inside this unit circle does belong to some disc: choose r to be the distance of that point to the origin to get an appropriate containing closed disc. We'll denote the open interior of this unit circle (the disc minus its boundary) by O_1.

Therefore $\bigcup_{D \in \mathcal{C}} D = O_1$.

Properties of Total Intersections and Unions

A number of properties that held for simple intersection and union also hold for arbitrary intersections and unions. We will consider the generalized *Distributive Laws* and *De Morgan's Laws*. Most of the proofs, along with other results, will be left as exercises.

PROPOSITION 4.2 - 1: Distributivity

a) $R \cap \left(\bigcup_{S \in \mathcal{C}} S \right) = \bigcup_{S \in \mathcal{C}} (R \cap S)$;

b) $R \cup \left(\bigcap_{S \in \mathcal{C}} S \right) = \bigcap_{S \in \mathcal{C}} (R \cup S)$.

***Proof*:**

a) We'll prove the first part and leave the second part for the exercises (see Exercise 17).

$$x \in R \cap \left(\bigcup_{S \in \mathcal{C}} S \right) \leftrightarrow x \in R \wedge (\exists S \in \mathcal{C})(x \in S)$$

$$\leftrightarrow (\exists S \in \mathcal{C})(x \in R \wedge x \in S)$$

$$\leftrightarrow (\exists S \in \mathcal{C})(x \in R \cap S)$$

$$\leftrightarrow x \in \bigcup_{S \in \mathcal{C}} (R \cap S). \quad \blacksquare$$

PROPOSITION* 4.2 - 2: *De Morgan's Laws

Let U be any set and let $\overline{S} = U - S$ be the complement of S relative to U.

a) $\overline{\bigcap_{S \in \mathcal{C}} S} = \bigcup_{S \in \mathcal{C}} \overline{S}$;

b) $\overline{\bigcup_{S \in \mathcal{C}} S} = \bigcap_{S \in \mathcal{C}} \overline{S}$.

***Proof*:**

See Exercise 18ab. $\blacksquare$

PROPOSITION* 4.2 - 3: *Intersections, Unions, and Subsets

a) $\bigcap_{S \in \mathcal{C}} S \subseteq T$ *for all* $T \in \mathcal{C}$;

b) $R \subseteq S$ *for all* $S \in \mathcal{C}$ *iff* $R \subseteq \bigcap_{S \in \mathcal{C}} S$;

c) $T \subseteq \bigcup_{S \in \mathcal{C}} S$ *for all* $T \in \mathcal{C}$;

d) $S \subseteq R$ *for all* $S \in \mathcal{C}$ *iff* $\bigcup_{S \in \mathcal{C}} S \subseteq R$.

***Proof*:**

b) We'll prove part b and leave the remaining parts as exercises (see Exercise 19abc).
First suppose R is a subset of every set S in the collection $\mathcal{C}$,
and let x be any element of R.
Then, since $R \subseteq S$, $x \in S$ for every set $S \in \mathcal{C}$.
This implies that $x \in \bigcap_{S \in \mathcal{C}} S$.
Therefore, $R \subseteq \bigcap_{S \in \mathcal{C}} S$.
Conversely, suppose $R \subseteq \bigcap_{S \in \mathcal{C}} S$.
Then, for any x in R, x lies in every S belonging to $\mathcal{C}$.
Thus $R \subseteq S$ for every S in $\mathcal{C}$. $\blacksquare$

To summarize this last proposition in words: parts a and b say that the intersection of a collection of sets is the largest set contained in each member of the collection; and parts c and d say that the union of a collection of sets is the smallest set containing each member of the collection. Thus, the intersection is the *least upper bound* of a collection of sets, ordered under the subset relation. The union is similarly the *greatest lower bound* of a collection of sets. These ideas will be explored later in connection with Boolean Algebra (see Section 7.2).

Partitions

Collections of sets often arise when a given set is partitioned into some number of subsets. A *partition* is a collection of pairwise disjoint subsets that all together exhaust the given superset. We will define this more formally after first defining what it means to be *pairwise disjoint*. This is a concept that plays an important role in some advanced mathematics courses, particularly analysis. Partitions are one of the reasons mathematicians are interested in collections of sets.

DEFINITION* 4.2 - 3: *Pairwise Disjoint Collections of Sets

A collection of sets $\mathcal{C}$ is pairwise disjoint iff $S \cap T = \emptyset$ for any two distinct sets S and T in the collection.

✠ **EXAMPLE 4.2 - 4**

Determine whether the collection of all open intervals of real numbers of the form $(n, n+1)$ is pairwise disjoint:

a) when $n \in \mathbb{Z}$;

b) when $n \in \mathbb{Q}$.

Solution

a) If only integer values of n are allowed, the collection of open intervals is pairwise disjoint. The nearest neighbors in the collection are then of the form $(n-1, n)$ and $(n, n+1)$, and these sets have no points in common.

b) However, if n is permitted to take on any rational number values, the collection of intervals is certainly not pairwise disjoint: for example, $(0,1) \cap (.5, 1.5) = (.5, 1)$.

Pairwise disjoint collections are strongly disjoint: this property asserts more than that the intersection of the collection as a whole is empty (see Exercises 20 – 22). Being pairwise disjoint is sometimes required in order for certain properties to hold. For instance, given a finite collection of finite sets, the total number of elements in the union is the sum of the individual numbers iff the collection is pairwise disjoint (see Section 4.5).

DEFINITION* 4.2 - 4: *Partition of a Set

A partition of a set S is a collection $\mathcal{C}$ of subsets of S which is pairwise disjoint and whose total union $\bigcup_{R \in \mathcal{C}} R$ is S.

✠ **EXAMPLE 4.2 - 5**

Does the collection of all open intervals of real numbers of the form $(n, n+1)$ form a partition of $\mathbb{R}$ if $n \in \mathbb{Z}$? if $n \in \mathbb{Q}$? Find a partition of $\mathbb{R}$.

Solution

Neither given collection forms a partition of $\mathbb{R}$.

The first one (when $n \in \mathbb{Z}$) doesn't because while it is pairwise disjoint, its union misses all integers and so doesn't yield $\mathbb{R}$.

The second collection also fails to be a partition, for while its union is all of $\mathbb{R}$, it is not pairwise disjoint (see Example 3).

Taking the collection of half-open/half-closed intervals $(n, n+1]$ for $n \in \mathbb{Z}$ forms a partition of $\mathbb{R}$: the collection is pairwise disjoint, and its union is all of $\mathbb{R}$.

Sets of Subsets: The Power Set

Given any set S, consider its subsets. Regardless of what S is, it has at least the extreme possibilities $\emptyset$ (nothing) and S (everything) as subsets. It will generally have many other subsets as well. The set consisting of all the subsets of S is called the *power set* of S and is denoted by $\mathcal{P}(S)$.

DEFINITION* 4.2 - 5: *Power Set
$\mathcal{P}(S) = \{R : R \subseteq S\}$.

The *power set operator* $\mathcal{P}$ is a strong unary operator. Given any set S, it collects all of its subsets into a set whose elements are those subsets. The power set operator obviously boosts us up onto a higher level of sets, generating large new sets of sets (see Exercise 31).

✠ **EXAMPLE 4.2 - 6**
Determine the power set for the set $S = \{1, 2, 3\}$.

Solution
The following eight subsets are the *elements* of $\mathcal{P}(S)$:
$\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$, $\{1, 2, 3\}$

The power set operator is monotone increasing, in the following precise sense. Given two sets, one of which is a subset of the other, their power sets are in the same subset relationship. This is the content of the next proposition. Related results exploring how the power set operator interacts with the operations of intersection and union are left for the exercises (see Exercises 32 – 37).

PROPOSITION* 4.2 - 4: *Subset Inclusion and Power Sets
$S \subseteq T \rightarrow \mathcal{P}(S) \subseteq \mathcal{P}(T)$

***Proof*:**
The proof here is trivial if you grab it in the right place. To see what to do, we'll use the *Method of Backward Proof Analysis*.

Assuming the antecedent of the given conditional as our supposition for *Conditional Proof*, we must prove the consequent. So we concentrate on that. It's a subset claim, which is proved by taking an element of the first set and showing that it also belongs to the second set. Now that we know how we want to start and where we want to go, we can begin to think about what the antecedent offers us. Focusing too early on what's given might confuse the proof by obscuring what needs to be proved.

So now suppose that $S \subseteq T$, and let $X \in \mathcal{P}(S)$.
Then $X \subseteq S$.
Since $S \subseteq T$, $X \subseteq T$, too.
Thus $X \in \mathcal{P}(T)$. ∎

EXERCISE SET 4.2

Problems 1 - 3: Pairing Sets

The following problems explore the notions of being a singleton and a doubleton.

1. Let $S = \{0\}$ and $T = \{0, 1\}$. What is $\{S, T\}$? Exhibit this set using only set-braces and the numbers 0 and 1.

*2. Let $S = \{0\}$ and $T = \{0, 1\}$.

a. Exhibit the set $R = T \cup \{T\}$, using only set-braces and the numbers 0 and 1.

b. Is $S \subseteq R$? Is $S \in R$? Explain.

c. Is $T \subseteq R$? Is $T \in R$? Explain.

3. Verify the claims made in the solution to Example 4.2-2b, that the doubleton $\{S, T\}$ differs from the sets S, T, $S \cap T$, $S \cup T$, and $S - T$ in the case where S is the set of even numbers and T is the set of odd numbers.

Problems 4 - 5: Finite Collections of Sets

Work the following problems on finite collections of sets.

*4. A collection $\mathcal{C}$ consists of the sets I_2, I_3, and I_4, where I_n denotes the set of all integers that are multiples of n.

*a. List the elements of I_2, I_3, and I_4.

*b. Determine $\bigcap_{S \in \mathcal{C}} S$.

c. Determine $\bigcup_{S \in \mathcal{C}} S$.

5. A collection $\mathcal{C}$ consists of the sets I_2, I_3, I_9, and I_{12}, where I_n denotes the set of all integers that are multiples of n.

a. List the elements of I_2, I_3, I_9, and I_{12}.

b. Determine $\bigcap_{S \in \mathcal{C}} S$.

c. Determine $\bigcup_{S \in \mathcal{C}} S$.

Problems 6 - 8: Plenty of Nothing

The following problems focus on distinguishing the empty set from collections that contain it.

*6. Explain why $\{\emptyset\} \neq \emptyset$.

*7. Explain why $\{\{\emptyset\}\}$ is different from both $\{\emptyset\}$ and $\emptyset$.

8. Explain why $\{\emptyset, \{\emptyset\}\}$ is different from $\emptyset$ and also from $\{\emptyset\}$ and $\{\{\emptyset\}\}$.

Problems 9 - 10: True or False

Are the following statements true or false? Explain your answer.

*9. $\{5, 6, \emptyset\}$ is a subset of $\{5, 6, 7\}$, since $\{5, 6\}$ is a subset of $\{5, 6, 7\}$ and $\emptyset$ is a subset of everything.

10. The basic properties that held for intersection and union of two sets also hold for any collection of sets.

Problems 11 - 12: Explanations

Explain the following terms/results in your own words.

11. Explain what a partition of a set is and give a concrete everyday example to illustrate it.

*12. Explain what the power set of a set is. If a given set S is a collection of people and subsets of S are considered committees formed from these people, what does $\mathcal{P}(S)$ represent?

Problems 13 - 14: Extending Set-Theoretic Definitions

The following problems deal with extending binary set operations to finitely many sets.

13. Use recursion to define the intersection $\bigcap_{i=1}^{n} S_i$ of finitely many sets S_i.

*14. Use recursion to define the union $\bigcup_{i=1}^{n} S_i$ of finitely many sets S_i.

Problems 15 - 16: Infinite Indexed Collections of Sets

The following problems use an indexed collection sets. The notation used is analogous to that of infinite series.

*15. For each i in $\mathbb{N}^+$, let $Q_i = \left(-\frac{1}{i}, \frac{1}{i}\right)$ be an open interval about 0 and $D_i = \left[-\frac{1}{i}, \frac{1}{i}\right]$ be the associated closed interval. Determine the following sets.

*a. $\bigcap_{i=1}^{\infty} Q_i$ *b. $\bigcup_{i=1}^{\infty} Q_i$

c. $\bigcap_{i=1}^{\infty} D_i$ d. $\bigcup_{i=1}^{\infty} D_i$

*16. For each i in $\mathbb{N}^+$, let $O_i = \left(-\frac{i}{i+1}, \frac{i}{i+1}\right)$ be an open interval about 0 and $C_i = \left[-\frac{i}{i+1}, \frac{i}{i+1}\right]$ be the associated closed interval. Determine the following sets.

a. $\bigcap_{i=1}^{\infty} O_i$ b. $\bigcup_{i=1}^{\infty} O_i$

*c. $\bigcap_{i=1}^{\infty} C_i$ *d. $\bigcup_{i=1}^{\infty} C_i$

Problems 17 - 19: Properties of Intersections and Unions

Prove the following propositions.

17. Prove *Proposition* 4.2-1b, that union distributes over intersection: $R \cup \left(\bigcap_{S \in \mathcal{C}} S\right) = \bigcap_{S \in \mathcal{C}} (R \cup S)$.

18. Prove *Proposition* 4.2-2, *De Morgan's Laws* for complements of total intersections.

a. $\overline{\bigcap_{S \in \mathcal{C}} S} = \bigcup_{S \in \mathcal{C}} \overline{S}$

b. $\overline{\bigcup_{S \in \mathcal{C}} S} = \bigcap_{S \in \mathcal{C}} \overline{S}$

19. Prove the following subset order properties from *Proposition* 4.2-3.

a. $\bigcap_{S \in \mathcal{C}} S \subseteq T$ for all $T \in \mathcal{C}$.

b. $T \subseteq \bigcup_{S \in \mathcal{C}} S$ for all $T \in \mathcal{C}$.

c. $S \subseteq R$ for all $S \in \mathcal{C}$ iff $\bigcup_{S \in \mathcal{C}} S \subseteq R$.

Problems 20 - 24: Pairwise Disjoint Sets

The following problems explore notions of disjoint sets.

*20. Prove that if a collection $\mathcal{C}$ of two or more sets is pairwise disjoint, then $\bigcap_{S \in \mathcal{C}} S = \emptyset$.

21. Is the converse to Problem 20 true or false? If it is true, prove it. If it is false, give a counterexample.

*22. Is it possible to find a collection $\mathcal{C}$ so that the intersection of every pair of distinct sets in $\mathcal{C}$ is nonempty while the total intersection of the collection is empty? Support your claim.

EC 23. Given a finite collection of distinct sets S_i for $i = 1, 2, \ldots, n$, show how to generate a new but related collection of sets D_i that has the same union as the original collection but is pairwise disjoint.

EC 24. Given an infinite collection of distinct sets S_i for $i \in \mathbb{N}$, is it possible to generate a collection of sets D_i that has the same union as the original collection but is pairwise disjoint? Why or why not?

Problems 25 - 29: Partitions

The following problems explore the idea of a partition.

25. Write out *Definition* 4.2-4 using the vocabulary of PL, indicating all quantifiers and logical connectives in their proper places.

*26. Let R_n denote all those natural numbers that leave remainder n when divided by 7 for $n = 0, 1, 2, \ldots, 6$. Explain why this collection of R_n is a partition of $\mathbb{N}$.

*27. Let $S_n = \{0, 1, \ldots, n\}$ denote the initial segment of $\mathbb{N}$ from 0 through n. Does the collection of all S_n form a partition of $\mathbb{N}$? Why or why not?

28. Let P_n be the set of all natural numbers that are powers of a prime number n. Does the collection of P_n for all prime numbers n form a partition of $\mathbb{N}$? Why or why not?

EC 29. Let S_i denote a collection of finite sets for $i = 1, 2, \ldots, n$, and let S be the total union of this collection. Define a new collection C_m by $C_m = \{x : x \text{ belongs to exactly } m \text{ sets of the original collection}\}$ for $m = 1, 2, \ldots, n$. Is the collection $\{C_m\}$ a partition of S or not? Explain.

Problems 30 - 31: Numerosity of the Power Set

The following problems concern the size of the power set $\mathcal{P}(S)$ of a set S.

*30. Determine $\mathcal{P}(S)$ for the following sets S.
- *a. $S = \{1\}$
- *b. $S = \{1, 2\}$
- c. $S = \{1, 2, 3, 4\}$

*31. *Numerosity of the Power Set*
- *a. Generalize Problem 30 and Example 6: if S has n elements, $\mathcal{P}(S)$ contains ______ elements. Prove your result using PMI.
- *b. How is the result you obtained in part a related to the alternative notation that is sometimes used to stand for the power set, namely, 2^S? Why do you think $\mathcal{P}(S)$ is called the *power set* of S?

Problems 32 - 37: Properties of Power Sets

Prove the following results on properties of power sets.

*32. Prove $\mathcal{P}(S \cap T) = \mathcal{P}(S) \cap \mathcal{P}(T)$.

33. Prove $\mathcal{P}\left(\bigcap_{S \in \mathcal{C}} S\right) = \bigcap_{S \in \mathcal{C}} \mathcal{P}(S)$.

34. Prove $\mathcal{P}(S \cup T) \supseteq \mathcal{P}(S) \cup \mathcal{P}(T)$.

35. Prove $\mathcal{P}\left(\bigcup_{S \in \mathcal{C}} S\right) \supseteq \bigcup_{S \in \mathcal{C}} \mathcal{P}(S)$.

36. Can the superset relation in Problems 34 − 35 be turned around? If so, prove it; if not, give a counter-example.

37. Can the conditional in *Proposition* 4.2-4 be turned around? If so, prove it; if not, give a counterexample.

EC 38. *Duality Principle for Set Theory?*
Several propositions have exhibited a sort of duality between intersection and union (see the remarks following the proof of *Proposition* 4.1-5). Formulate a *Duality Principle* for *Set Theory* and then explore the truth of your statement by verifying it or refuting it in a variety of specific instances. If you arrive at a final formulation of the principle that you think is true, give an argument to justify it.

HINTS TO STARRED EXERCISES 4.2

2. a. Start with what R is and replace T by what it is equal to.
   b. Keep straight that $S = \{0\}$; $S \neq 0$.
   c. For this part it will be simpler to use the original definition of R and T instead of your answer to part a.

4. a. [No hint.]
   b. This intersection is the same as $I_2 \cap I_3 \cap I_4$.

6. For two sets to be equal, each must contain the elements of the other.

7. Think of sets as collectors for their elements, and recall when two sets are equal.

9. [No hint.]

12. [No hint.]

14. Recursion needs an initialization step ($n = 1$) and an induction step (passing from $n = k$ to $n = k + 1$).

15. [No hint.]

16. [No hint.]

20. Use a *CP* proof strategy; review the definition of intersection to see what needs proving.

22. No fancy collection is needed here for a counterexample. Three small sets will do.

26. This is similar to Example 1. Review the requirements for being a partition.

27. Review the requirements for being a partition, and check whether they are met by this collection.

30. See Example 6.

31. a. Find a formula involving n. The induction argument should start at $n = 0$. For the induction step, temporarily remove some element to help you make the necessary connection.
    b. Look at the formula you obtained in part a.

32. Use your knowledge of when two sets are equal and the definition of the power set. You should be able to string iff-statements together here.

4.3 Multiplicative Counting Principles

Sections 4.1 and 4.2 laid the basic set-theoretic groundwork for several themes that we will be exploring in this text. In the rest of this chapter we will look at two main topics, both closely connected with numerosity; i.e., with the number of elements belonging to a set.

We will begin by considering some elementary topics in *combinatorics*, learning how to enumerate possibilities in a variety of additive and multiplicative situations involving arrangements or combinations. This will be our focus in this section and the next two.

In the next chapter we will investigate the issue of numerosity a bit more theoretically, looking at some issues related to infinite sets. This ties in with how *Set Theory* originated; it has both technical and philosophical dimensions that are relevant to mathematics and computer science.

Ordered Pairs and Cartesian Product of Sets

The final binary set-theoretic operation we will consider is *Cartesian product*. This operation will enable us to treat relations and functions as an integral part of *Set Theory*. It is also an important pillar of our treatment of combinatorics.

The Cartesian-product operator takes two sets S and T and forms the set $S \times T$ of all possible ordered pairs from them. We'll introduce this notion via an example and then give the formal definition.

✠ **EXAMPLE 4.3 - 1**

A not-very-style-conscious mathematics professor owns eight different shirts (call them $S_1, S_2, \ldots, S_8$) and six different trousers (label them $T_1, T_2, \ldots, T_6$). If his wife weren't around, he would no doubt end up wearing any shirt with any pair of pants. Use the notion of ordered pairs and Cartesian product to indicate all the potential outfits this prof might show up wearing to class.

Solution

Each ordered pair (S_i, T_j) represents one outfit that might be worn. The set $S \times T$ of all such ordered pairs $\{(S_i, T_j) : 1 \leq i \leq 8, 1 \leq j \leq 6\}$ gives the collection of all possible outfits. This gives a total of 48 different outfits, some of them probably pretty poorly matched.

DEFINITION* 4.3 - 1: *Cartesian Product

$S \times T = \{(x, y) : x \in S \wedge y \in T\}$.

Our definition of Cartesian product assumes the idea of an ordered pair. This seems reasonable; everyone is familiar with ordered pairs from graphing points and functions in elementary algebra. In a rigorous systematic development of this topic, however, the Cartesian-product operator would be more thoroughly grounded in *Set Theory* by defining ordered pairs in terms of sets. Since this would introduce a higher order of abstractness into the discussion, we will leave it for the exercises (see Exercises 17 – 19).

Cartesian products cannot be pictured by means of Venn diagrams, but they can often be graphed nevertheless, as the following example illustrates.

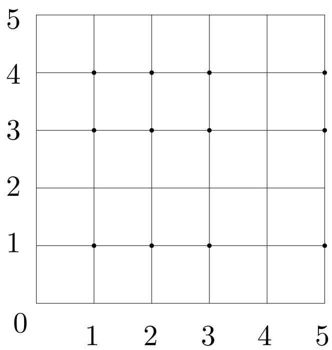
✠ **EXAMPLE 4.3 - 2**

a) If $S = \{1, 2, 3, 5\}$ and $T = \{1, 3, 4\}$, graph the Cartesian product $S \times T$.
b) If $S = [1, 5]$ and $T = [1, 4]$, graph the Cartesian product $S \times T$.

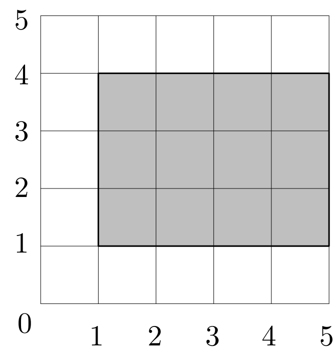
Solution

a) $S \times T = \{(1,1),(1,3),(1,4),(2,1),(2,3),(2,4),(3,1),(3,3),(3,4),(5,1),(5,3),(5,4)\}$. This can be plotted as a set of 12 distinct points in a coordinate grid, as illustrated below.

b) For S being the entire closed interval of real numbers $[1,5]$ and T being the closed interval $[1,4]$, $S \times T$ consists of all possible points on or inside the rectangular region $\{(x,y) : 1 \leq x \leq 5; 1 \leq y \leq 4\}$. This is also plotted below, as the shaded region.



$S \times T$



$[1,5] \times [1,4]$

It is possible to explore how the operation of Cartesian product interacts with the other ones already given. A number of these properties are given in the exercises (see Exercises 29 – 39). We will have little occasion to use them in the remainder of this book, but working them will help you become better acquainted with the idea of a Cartesian product as well as the other operations.

Besides the notions of ordered pairs and Cartesian products, we will want to use their generalizations: *ordered n-tuples* $(x_1, x_2, \ldots, x_n)$ and *n-fold Cartesian products* $S_1 \times S_2 \times \cdots \times S_n$ and $S^n = S \times S \times \cdots \times S$. As above, we will assume the notion of an ordered n-tuple as given (see Exercises 20 – 23 for a more explicit definition), and we will use it to define finite Cartesian products. The related notion of a *finite sequence* is also defined using these ideas.

DEFINITION* 4.3 - 2: *Finite Cartesian Products

a) $S_1 \times S_2 \times \cdots \times S_n = \{(x_1, x_2, \ldots, x_n) : x_i \in S_i\}$.

b) $S^n = \{(x_1, x_2, \ldots, x_n) : x_i \in S\}$.

DEFINITION* 4.3 - 3: *Finite Sequences

a) *A finite sequence $(x_1, x_2, \ldots, x_n)$ of length n selected from a set S is an element of S^n.*

b) *A finite sequence without repetition selected from a set S is a finite sequence in which no element of S appears more than once: $x_i \neq x_j$ if $i \neq j$.*

Multiplicative Counting Principle

If a first action can be done in m ways and for each of these m ways a second action can be done in n ways, then the total compound action — first action, second action — can be done in $m \times n$ ways. We already saw this to be the case in Example 1: there were $8 \times 6 = 48$ different outfits that could be made from the pants and shirts available. This *Multiplicative Counting Principle* follows from the fact that the cardinality (numerosity) of the Cartesian product of two sets is the product of the cardinalities of the sets (see *Proposition* 1 below).

DEFINITION* 4.3 - 4: *Cardinality of a Set

The cardinality of a set S, denoted by $|S|$, is the number of elements contained in S.

This definition basically defines cardinality in terms of numerosity, which is merely a better known synonym. Thus, the cardinality of a set S tells *how many* members it has. We will associate a cardinality, a number size, with each set S, whether it is finite or infinite. What sense this makes for infinite sets will be taken up later. For the rest of this chapter, however, we will assume a context of finite sets, where the concept is intuitively clear.

The following proposition uses $\times$ in two ways: as the set-theoretic notation for Cartesian product, and as the ordinary multiplication symbol for numbers. The connection asserted by this proposition is the motivation for why $\times$ gets used for the Cartesian product operator.

PROPOSITION* 4.3 - 1: *Cardinality of Cartesian Products

a) $|S \times T| = |S| \times |T|$

b) $|S_1 \times S_2 \times \cdots \times S_n| = |S_1| \times |S_2| \times \cdots \times |S_n|$

c) $|S^n| = |S|^n$

***Proof*:**

See Exercises 24 – 25. ■

COROLLARY* 1: *Multiplicative Counting Principle

If one choice can be made in m ways and for each of these choices a second choice can be made in n ways, then the combined joint choice can be made in $m \times n$ ways.

***Proof*:**

Model the choices using ordered pairs (first choice, second choice) from two sets F and S. Then the set of all the different combined choices is the associated Cartesian product $F \times S$. *Proposition* 1a yields the result claimed. ■

The last result can be generalized to any finite sequence of component choices. This yields the following corollary.

COROLLARY* 2: *Generalized Multiplicative Counting Principle

If each of k choices can be made in n_i ways for $i = 1, 2, \ldots, k$, then the total number of distinct choice sequences of length k is $\prod_{i=1}^{k} n_i = n_1 \cdot n_2 \cdots n_k$.

***Proof*:**

This follows immediately from *Proposition* 4.3-1b or by applying mathematical induction to *Corollary* 1 (see Exercise 26). ■

The *Multiplicative Counting Principle*, in either of its two forms, lies behind several different counting techniques. We'll explore some of these in the rest of this section and some more in Section 4.4. We will introduce the next one with an example.

✠ **EXAMPLE 4.3 - 3**

A *byte* is an 8-bit string, such as 01010101, each *bit* (binary digit) being either a 0 or a 1.

a) How many distinct bytes are possible?

b) How many of them begin or end with four 0s?

Solution

a) Since each bit has 2 possibilities, 0 or 1, and each one can be chosen independently of the others, an eight-bit string can be formed in $2^8 = 256$ ways.

b) If a byte begins with four 0s, there are $2^4 = 16$ different ways to finish the byte. This is also the total number of such bytes.
If a byte ends with four 0s, there are 16 ways to begin the byte, and so again there are 16 different bytes.
The only byte counted as a part of both sets is the one that starts and ends with four 0s; i.e., the byte with all zeros. Counting this only once, we have $16 + 15 = 31$ bytes that start or end with four zeros.

Ordered Choice With Repetition

Example 3 can be thought of as creating choice sequences of length eight by selecting each component bit from the same sample space of two available possibilities, the set $\{0, 1\}$. Each choice is permissible, regardless of how the last choice was made; both 0 and 1 can be used repeatedly. This is an example of an *Ordered Choice with Repetition* taken from a common sample space.

PROPOSITION* 4.3 - 2: *Counting Ordered Choices with Repetition

Suppose S is a sample space of n elements. Then the total number of choice sequences of length k with elements selected from S, allowing repetition, is n^k.

***Proof*:**

This is a simple application of the *Generalized Multiplicative Counting Principle* with each $n_i = n$. ■

✠ **EXAMPLE 4.3 - 4**

An IA license plate has three letters followed by three numbers. If any letters and any numbers can be used, how many distinct IA license plates can be made?

Solution

This requires a combination of the counting methods we've introduced so far.
By the method for counting ordered choices with repetition, there are 26^3 ways that three-letter words can be made, and there are 10^3 three-digit numbers (we include 000).
Thus, using the *Multiplicative Counting Principle*, there are $26^3 \cdot 10^3 = 17,576,000$ different license plates possible. Since there are around $3,107,000$ people in IA (and pigs and cows can't drive), this number is quite adequate for IA, even if everyone owned a few vehicles.

Permutations: Ordered Choice Without Repetition

Let's now consider the case where the sample space is gradually being depleted each time a choice is made, so that no repetition of choice is allowed. Suppose, moreover, that order remains important: think of the choice as being made sequentially, as before. Counting the number of total choice sequences of a given length can still be done via the *Generalized Multiplicative Counting Principle*.

✠ **EXAMPLE 4.3 - 5**

In a cross-country race, the places of the first five runners of each team to cross the finish line are counted toward the scoring. If a team has 12 runners entered in a race, in how many different ways might these players contribute toward the score of their team?

Solution

The number of different ways runners can potentially finish in the first five places for their team is $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 = 95{,}040$. Note that if this number were both multiplied and divided by 7!, we would have our answer in the more compact form $12!/7!$.

If you are calculating the number of choice sequences without repetition for fairly short sequences (as in the last example), it is probably easiest to multiply the k factors together. But the factorial formula indicated there is better for long sequences. It also comes in handy when you're using such a result in a computer program or calculating the number by hand, since you can then make use of the built-in factorial function on your calculator or computer.

We will now provide the theory behind the counting process exhibited in this example. *Proposition* 4.3-3 summarizes the outcome, but we first introduce some standard terminology.

A *permutation* of a set S is an ordered arrangement of all its elements. Arrangements can be thought of as arising by selecting elements of a set in succession until it is exhausted. Each permutation is thus uniquely associated with ordered choice sequences in which all the elements of the sample space appear exactly once, with a choice sequence without repetition of length n, where $n = |S|$.

If k distinct things from the full sample space S are arranged without repetition, we have a *k-permutation* of S. Such arrangements are uniquely associated with choice sequences without repetition of length k. A k-permutation in which $k = |S|$ is thus just an ordinary permutation of S.

Given this association between arrangements and choice sequences, we can formally define permutations as follows.

DEFINITION* 4.3 - 5: *Permutations

a) *A permutation of a set S with $|S| = n$ is an ordered n-tuple $(x_1, x_2, \ldots, x_n)$ with $x_i \neq x_j$ for $i \neq j$.*

b) *A k-permutation of a set S with $|S| = n$ is an ordered k-tuple $(x_1, x_2, \ldots, x_k)$ with $x_i \neq x_j$ for $i \neq j$.*

PROPOSITION* 4.3 - 3: *Counting k-Permutations/Ordered Choices without Repetition

Suppose S is a sample space of n elements. Then the total number of k-permutations of S is given by $P(n, k) = n \cdot (n-1) \cdots (n-(k-1)) = \dfrac{n!}{(n-k)!}$.

***Proof*:**

From the *Generalized Multiplicative Counting Principle*, the number of choice sequences of length k from S, not allowing repetition, is $n \cdot (n-1) \cdots (n-(k-1))$. This is thus the number of k-permutations of S.

Multiplying and dividing by $(n-k)!$ gives the result in factorial format. ■

COROLLARY*: *Counting Permutations

The total number of distinct permutations of a set S is $n!$, where $n = |S|$.

***Proof*:**

This is an easy corollary of the last proposition, taking $k = n$. (Recall that $0! = 1$.) ■

✠ **EXAMPLE 4.3 - 6**

A quiz has 10 *matching* questions on it with 10 possible answers. If it is answered randomly by someone who forgot to study the material, how many different quizzes can be turned in, assuming each answer is used exactly once.

Solution

There are $10! = 3,628,800$ different ways this quiz can be filled in. Presumably, only one of them is correct, so pure guessing wouldn't be a very high-percentage strategy.

✠ **EXAMPLE 4.3 - 7**

Call any character string formed by the first seven letters of the alphabet a,b, ... , g a full scrabble segment.

a) How many full scrabble segments are there?
b) How many full scrabble segments have the vowels a and e next to each other?
c) How many full scrabble segments have the vowels separated from each other?
d) Are there any full scrabble segments that make a real word (called a *bingo*)?

Solution

a) There are $7! = 5040$ full scrabble segments: our sample space is $S = \{a, b, c, d, e, f, g\}$, and we are counting its permutations.
b) In order to work this problem, think of a and e as forming a vowel-block, and consider each of the other letters as individual consonant blocks.
   We must now choose six blocks in succession (our sample space consists of these six blocks now, not letters) without repetition. There are $6! = 720$ of these block-sequences. Since the vowel block can appear either as ae or ea, the joint choice (block sequence, vowel arrangement) yielding our full scrabble segment can be done in $720 \cdot 2 = 1440$ ways.
c) If 1440 of the 5040 full scrabble segments have the a and e next to one another, the other $5040 - 1440 = 3600$ full scrabble segments must not.
d) Here we're going beyond what mathematics can decide. According to a Scrabble expert I consulted, no full scrabble segment forms a genuine word.

EXERCISE SET 4.3

Problems 1 - 3: Cartesian Products

Determine the following Cartesian products.

*1. Write out the elements of the Cartesian product $E \times P$, where E is the set of positive even integers less than 10 and P is the set of primes less than 10. Then graph this set in a coordinate grid.

2. Determine the Cartesian product of $\mathbb{Z} \times \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers. Then graph this set. (This is called the set of *integer lattice points*.)

3. What does the graph of $S \times S$ look like, where S is the set of non–negative real numbers?

Problems 4 - 6: Scrabble Segments

The following problems deal with scrabble segments (see Example 7 above for a definition).

4. How many full scrabble segments have all five consonants together?

*5. How many full scrabble segments have the letters a and e separated by the letter c?

*6. How many full scrabble segments have the vowels a and e separated by one consonant? by two consonants?

Problems 7 - 11: Palindromes

A ***palindrome*** *is any number that reads the same way forward and backward, such as* 54321012345*. Repetition is permitted, but no such number is written with a leading* 0.

*7. How many seven-digit palindromes are there? How many of them are even numbers?

8. How many eight-digit palindromes are there? How many of them are odd numbers?

*9. How many palindromes are there of length $2n+1$?

10. How many palindromes are there of length $2n$?

EC 11. Find a formula that gives the total number of palindromes of length n.

Problems 12 - 14: Counting Divisors

The following problems have to do with the numbers of factors a number has.

*12. *Factors of* 60
   a. List and count the number of distinct divisors of 60 (include both 1 and 60).
   b. Factor 60 into a product of powers of primes. How do the prime factors of divisors of 60 relate to the prime factors of 60?
   c. Using your result in part *b* and the methods of this section, count the total number of distinct divisors of 60.

13. *Factors of* 72
   a. List and count the number of distinct divisors of 72 (include both 1 and 72).
   b. Factor 72 into a product of powers of primes. How do the prime factors of divisors of 72 relate to the prime factors of 72?
   c. Using your result in part *b* and the methods of this section, count the total number of distinct divisors of 72.

14. *Numbers of Factors and Prime Factorization*
   a. If $n = p \cdot q$, where p and q are both prime numbers, how many factors will n have? Explain.
   b. If $n = p^k \cdot q^m$, where p and q are prime numbers, how many factors will n have? Explain.
   EC c. If $n = p_1^{n_1} \cdot p_2^{n_2} \cdots p_k^{n_k}$ where all p_i are prime numbers, how many distinct divisors does n have? Explain using the methods of this section.

Problems 15 - 16: True or False

Are the following statements true or false? Explain your answer.

*15. Let S be any set with $T = \{1\}$. Then $S \subseteq S \times T$.

16. If $S = \{2,4,6\}$ and $T = \{3,5,7\}$, then $SxT = \{2*3, 4*5, 6*7\} = \{6,20,42\}$.

Problems 17 - 19: Rigorously Defining Ordered Pairs

The following unusual but ingenious definition of ordered pair is due to Norbert Wiener (1914) as simplified by Kasimierz Kuratowski (1921). It allows ordered pairs to be treated as special kinds of ordinary (non-ordered) sets and so be incorporated into Set Theory as a derivative notion.

***Definition of Ordered Pair*:** $(x,y) = \{\{x\},\{x,y\}\}$.

*17. Write the set-theoretic representation for the ordered pair $(0,1)$.

*18. What ordered pair does $\{\{2,3\},\{3\}\}$ represent?

EC 19. Using the above definition and given your intuitions about singletons and doubletons, when will $(a,b) = (c,d)$? Give a proof of your result.

Problems 20 - 23: Defining Ordered n-tuples

Ordered n-tuples can be defined recursively, using ordered pairs as a basis. The recursive clause is as follows:

***Definition of Ordered n-tuple*:** $(x_1, x_2, \ldots, x_n, x_{n+1}) = ((x_1, x_2, \ldots, x_n), x_{n+1})$.

20. *Ordered Triples*
   a. Determine what an ordered triple (a,b,c) is in terms of ordered pairs.
   b. Determine what an ordered triple (a,b,c) is in its most primitive form, using the Wiener-Kuratowski definition of ordered pair to reduce it as far as possible (see above).

*21. *Ordered Quadruples*
Write down the definition for a 4-tuple (a, b, c, d) and then work it backwards to express it in terms of ordered pairs.

22. *Ordered Quintuples*
Based on the pattern of Problems 20 - 21, what do you think (a, b, c, d, e) is expressed in terms of ordered pairs? Prove your conjecture by working it down into its primitive form via the recursive definition.

23. If the above definition is the inductive clause of the definition, what is the initialization clause? Why can't the initialization clause begin with $n = 1$?

Problems 24 - 26: Cardinality of Cartesian Products

Prove the following results about the cardinality of Cartesian products.

*24. Prove *Proposition* 4.3-1a: If $|S| = m$ and $|T| = n$, then $|S \times T| = m \times n$. Hint: what main proof techniques do you have available to show that a result holds for all natural numbers m and n? Use a combination of both direct approaches.

25. Use *PMI* and *Proposition* 4.3-1a (Problem 24) to prove *Proposition* 4.3-1b: $|S_1 \times S_2 \times \cdots \times S_n| = |S_1| \times |S_2| \times \cdots \times |S_n|$.

26. Use mathematical induction to prove *Corollary* 2 to *Proposition* 4.3-1: *If each of k choices can be made in n_i ways, then the total number of distinct choice sequences of length k is $n_1 \cdot n_2 \cdots n_k$.*

Problems 27 - 28: Strings and Finite Sequences

Strings $a_1a_2 \cdots a_k$ of length k can be considered finite sequences of length k (though written without parentheses or commas), each entry a_i coming from some common alphabet set A. The set of all such k-strings is denoted by A^k. $A^* = \bigcup_{k=1}^{\infty} A^k$ is thus the set of all finite strings formed from the alphabet.

27. Let A be the English alphabet, with $|A| = 26$.
   a. How many strings are there of size 5? How many 5-strings are there if no repetition is permitted?
   b. What does A^* represent in this case?

*28. Let A be the set of digits $0, 1, \ldots, 9$.
   a. How many strings are there of size 6? How many are there if the first entry is non-0?
   b. What does A^* represent in this case?

Problems 29 - 39: Theorems About Cartesian Products?

Determine whether the following results about Cartesian products are theorems of Set Theory or not. If they are true, illustrate them via an appropriate diagram and then prove them; if they are not, provide a specific counterexample. If you see an obvious way to fix a false result, restate it and then prove it.

29. $\emptyset \times S = \emptyset = S \times \emptyset$

30. $S \times T = \emptyset \leftrightarrow S = \emptyset \vee T = \emptyset$

*31. $S \times T = T \times S$

32. $S_1 \subseteq S_2 \wedge T_1 \subseteq T_2 \leftrightarrow S_1 \times T_1 \subseteq S_2 \times T_2$

33. $R \times (S \cap T) = (R \times S) \cap (R \times T)$

*34. $R \times (S \cup T) = (R \times S) \cup (R \times T)$

35. $R \times (T - S) = (R \times T) - (R \times S)$

36. $(S_1 \times T_1) \cap (S_2 \times T_2) = (S_1 \cap S_2) \times (T_1 \cap T_2)$

37. $(S_1 \times T_1) \cup (S_2 \times T_2) = (S_1 \cup S_2) \times (T_1 \cup T_2)$

38. $(S_1 \times T_1) - (S_2 \times T_2) = (S_1 - S_2) \times (T_1 - T_2)$

EC 39. $\overline{S \times T} = \overline{S} \times \overline{T}$

HINTS TO STARRED EXERCISES 4.3

1. See Example 2. Also, recall that 1 is not prime.

5. Consider the three mentioned letters as one block and each of the remaining letters as their own block. Don't forget to consider all the ways the big block can be rearranged.

6. Use the same method that you applied to Exercise 5.

7. In counting these, note that the last three numbers of the palindrome are completely determined by the first three.

9. Generalize what you did in Exercise 7.

12. a. [No hint.]
    b. Compare how each divisor of 60 can be factored to the way 60 was factored.
    c. Each divisor of 60 has 0, 1, or 2 factors of 2; etc. Use this to count the total number of divisors.

15. [No hint.]

17. Replace the x's in the definition with 0 and the y's with 1.

18. Recall that sets are non-ordered. Thus $\{\{2,3\},\{3\}\} = \{\{3\},\{2,3\}\}$, etc.

21. Rewrite the ordered quadruple as the ordered pair containing an ordered triple and a single element, then break down the ordered triple similarly.

24. Review the recursive definition of multiplication of $m \cdot n$ (Definition 3.3-4). Which of the two numbers does the induction clause work with? Use *PMI* on that number here.

28. a. Think in terms of the numbers being represented.
    b. If you only had strings starting with a non-0 digit, what would A^* be? How does allowing a beginning 0 change A^*?

31. Two well-chosen small sets will give a counterexample here. It would be a good exercise to prove in general that the cardinality of these two sets, however, are equal.

34. Use a string of iffs to argue the set membership statement needed, making use of an SL *Replacement Rule*.

4.4 Combinations

We're now able to count the number of ways in which a compound action can occur, provided order is important — both when a repeated choice is allowed and when it isn't (*permutations*). We modeled this situation set-theoretically via Cartesian products and finite sequences of different sorts, and we found ways to count the possibilities based ultimately on the *Multiplicative Counting Principle*.

In this section we will focus on *combinations*, compound events where order is irrelevant. We will model this situation not with finite sequences taken from some sample space, but with finite subsets of a sample space. To count them, we must develop a method of determining the cardinality of subsets.

We will look at several applications of this topic. One important mathematical application is that of evaluating binomial coefficients in the expansion of $(a+b)^n$; these coefficients are the numbers found in *Pascal's Triangle*. Counting permutations and combinations also forms an important foundation for discrete probability theory. We will introduce this topic here, but barely: lots more can be done with this than we have time to spend on it.

Combinations: Unordered Choice Without Repetition

Suppose we have a sample space S of n different elements to choose from, and we pick k of them, with no concern for the order in which they are chosen and without allowing repetition. Think of choosing them all at once instead of sequentially. In how many different ways can this activity be done?

Essentially, this question asks: how many different subsets of size k does a set of size n have? Distinct combinations are associated with distinct subsets. This number is denoted by one of three notations: by $C(n,k)$ or ${}_nC_k$, which indicates the number of k-combinations from a set of size n; or by $\binom{n}{k}$, which is read as "n choose k" or as "n binomial k" (see below).

We'll answer this question by building on what we did in Section 4.3 with ordered choice. We will first count the number of ordered combinations (permutations), and then we will divide out by the duplication number introduced by the ordering. The same set of elements can appear in numerous ways if order is taken into account, but we should count these only once if order is to be neglected.

An ordered set of k elements, without repetition, is a k-permutation. For a sample space S with $|S| = n$, there are $P(n,k) = n \cdot (n-1) \cdots (n-(k-1)) = \dfrac{n!}{(n-k)!}$ such permutations. Each distinct permutation belongs to a set of $k!$ permutations, all having the same elements but in different orders. To count this as a single combination, we must divide our total by the multiplicity involved. This yields the following proposition.

PROPOSITION* 4.4 - 1: *Counting Combinations without Repetition

Let S be a sample space of n elements. Then the total number of subsets of S of size k is given by $C(n,k) = \dbinom{n}{k} = \dfrac{n \cdot (n-1) \cdots (n-(k-1))}{k!} = \dfrac{n!}{k!\,(n-k)!}$.

An easy way to remember these formulas is to note that in the first expression the numerator and denominator have the same number of k factors, one going down from n, the other coming up from 1. In the second expression, the values whose factorials are being multiplied in the denominator (k, $n-k$) add up to the number whose factorial is being taken in the numerator (n).

✠ **EXAMPLE 4.4 - 1**

a) A math prof has 7 whiteboard markers sitting on her desk. If she takes 3 of them to class, in how many different ways can she do this?

b) How many subsets of size 3 does a set of size 7 have? How many subsets of size 4?

Solution

a) This problem asks how many different combination of 3 markers can be chosen from a sample space of 7 markers: it's a combination-without-repetition problem.
The total number is thus $\binom{7}{3} = \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} = \frac{7!}{3!\,4!} = 35$ distinct trios of markers.

b) There are 35 distinct subsets of size 3 in a set of size 7; this is just the abstract version of the last part.
There are also 35 distinct subsets of size 4. This is not a coincidence. For example, every choice of 3 markers in part a leaves a corresponding set of 4 markers in the prof's office. Furthermore, the formula for the total number of subsets of size k is exactly the same as that for subsets of size $n-k$: $\binom{n}{k} = \binom{n}{n-k}$. This makes sense in terms of the subsets, as just noted: associated with each subset of size k there is a unique subset of size $n-k$; namely, its complement inside the sample space.

Combinations, Binomial Coefficients, and Pascal's Triangle

The method of counting combinations-without-repetition can be used to determine the coefficients appearing in the expansion of the binomial $(a+b)^n$ for a positive integer n. For instance, $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. Calculating $(a+b)^n$ as an n-fold product gives the following theorem. This result is generalized to other sorts of exponents in introductory calculus.

THEOREM* 4.4 - 1: *Binomial Expansion Theorem

Let a and b be real numbers and n be a natural number. Then $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$.

***Proof*:**

Since $(a+b)^n = (a+b)(a+b)\cdots(a+b)$, the different terms in the expansion arise by choosing one factor from each binomial expression (either a or b) and then multiplying them together to get an n-fold product. We will focus on the ways to form the powers b^k.
Choosing no b's gives a^n; there is only one way this occurs. This can be done in exactly $\binom{n}{0} = 1$ way, giving a^n.

Choosing one b and all a's for the other factors, the term $a^{n-1}b$ arises; this can be done in $\binom{n}{1} = n$ ways; this yields the term $na^{n-1}b$.

Similarly, the coefficient of $a^{n-k}b^k$ is $\binom{n}{k}$ for every k; that of b^n is $\binom{n}{n} = 1$.

Thus, the entire binomial expansion can be written as $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$. ■

The binomial coefficients can be put into a triangular array known as Pascal's triangle.* Row n of the triangle gives the sequence $\left\{\binom{n}{k}\right\}_{k=0}^{n}$, starting with $n=0$ as the top row.

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* This is named after the seventeenth-century French mathematician Blaise Pascal, who investigated its properties. It was known several centuries earlier, however, both to Arabic mathematicians and Chinese mathematicians.

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1 \\
 & & & & \vdots & & & &
\end{array}
$$

Pascal's Triangle: Binomial Coefficients

Pascal's triangle can be generated recursively from the top two rows. Each later row begins and ends with a 1, and all intermediate numbers are generated by adding the two adjacent numbers directly above them. In the last row exhibited, $4 = 1 + 3$, $6 = 3 + 3$, and $4 = 3 + 1$. Pascal's triangle is a favorite for finding patterns of numbers. Some of these are explored in the exercises (see Exercises 18 – 25).

✠ **EXAMPLE 4.4 - 2**

Expand $(1 + 1)^n$ as a binomial and explain its meaning for the numerosity of $\mathcal{P}(S)$.

Solution

On the one hand, $(1 + 1)^n = 2^n$.

On the other hand, $(1+1)^n = \sum_{k=0}^{n} \binom{n}{k} 1^{n-k} 1^k = \sum_{k=0}^{n} \binom{n}{k}$.

So $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

If S is a set of size n, it has $\binom{n}{k}$ subsets of size k. Therefore, $\sum_{k=0}^{n} \binom{n}{k}$ represents the total number of subsets of S.

Thus, the cardinality of $\mathcal{P}(S)$ is 2^n.

Permutations, Combinations, and Discrete Probability

The probability of an event is a measure of how likely it is to occur. Being able to count numbers of permutations and combinations makes it possible to calculate the theoretical probability of certain events. We will state the classical relative-frequency definition for discrete probability and then look at a few fairly standard examples.

DEFINITION* 4.4 - 1: *Discrete Probability

The probability that a certain event will occur is the relative frequency with which it can occur: i.e., the ratio of the number of favorable ways the event can occur to the total number of possible outcomes.

This definition assumes that each possibility in the sample space S (the set of all possible outcomes) is equally likely. Under this assumption, the probability of an event E occurring is given by the formula $P(E) = \frac{|E|}{|S|}$. Note that the probability of an event occurring satisfies the double inequality $0 \leq P(E) \leq 1$ since $\emptyset \subseteq E \subseteq S$.

✠ **EXAMPLE 4.4 - 3**

A fair coin is tossed five times.

a) What is the probability that exactly three heads will occur?

b) What is the probability that exactly three tosses land the same way?

Solution

a) Take the sample space S to be all 5-tuples of H's and T's (potential toss outcomes). There are $2^5 = 32$ outcomes possible, all of them equally likely if the coin is fair.
To get exactly 3 heads, this must happen on 3 specific tosses. The number of ways 3 tosses can be chosen out of 5 is $\binom{5}{3} = 10$ ways.
Thus the probability of getting exactly three heads is $10/32 = 5/16 = .3125$.

b) If exactly three tosses land the same way, these can be either heads or tails. There are 10 ways for each of these to occur. So the probability of this event is $20/32 = 5/8 = .625$.
Note that since these two sub-events are disjoint (3 heads; 3 tails), their individual probabilities add up to the total probability: $5/16 + 5/16 = 5/8$.

✠ EXAMPLE 4.4 - 4

Two dice are rolled. What is the probability of getting a 7? An 11? A 7 or an 11?

Solution

Each die has six possible outcomes, so tossing a pair of dice yields 36 distinct pairs of numbers.
In order to get a sum of 7, the numbers must be $1 + 6$ or $2 + 5$ or $3 + 4$. Each possibility can occur in two ways, so there are 6 ways a 7 can be rolled. This gives a probability of $6/36 = 1/6 = .1\overline{6}$ for rolling a 7.
Rolling an 11 is less frequent: it only comes about as $5 + 6$, which can happen in 2 ways. The probability of throwing an 11, therefore, is $2/36 = 1/18 = .0\overline{5}$.
Thus, the probability of rolling either a 7 or an 11 is $8/36 = 2/9 = .\overline{2}$.

✠ EXAMPLE 4.4 - 5

Given a standard 52-card deck (four suits with 13 different kinds of cards), which five-card hand has a higher probability: a full house (three of one kind, two of another) or a flush (five in the same suit)?

Solution

- Our sample space here consists of all possible 5-card hands.
  There are $\binom{52}{5} = \frac{52!}{5!\,47!} = 2,598,960$ hands in all.
- The number of ways to get a full house is calculated by multiplying the number of ways to choose one kind times the number of ways to get three of this kind times the number of ways to choose a second kind times the number of ways to get two of this other kind.
  This number is $13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2} = 156 \cdot 4 \cdot 6 = 3744$.
  (Note: the product $\binom{13}{2} \cdot \binom{4}{3} \cdot \binom{4}{2}$ is off by a factor of 2 because it doesn't take into account which face-value is three of a kind and which one is two of a kind.)
- A flush can be generated by first choosing a suit and then choosing 5 cards from that suit.
  The number of ways in which this can occur is thus $4 \cdot \binom{13}{5} = 4 \cdot 1287 = 5148$.
- Thus, a flush is slightly more likely than a full house: its probability is $5148/2,598,960 \approx .00198$, compared with the probability of $3744/2,598,960 \approx .00144$ for a full house. A flush is likely to occur about 54 more times than a full house in 100,000 hands.

Unordered Choice with Repetition

The most complex combinatorial situation is that of counting the number of unordered samples when repetition is allowed. The difficulty arises here in part because such samples no longer correspond to subsets of a sample space, since sets cannot represent an element's being present multiple times. Furthermore, while we could begin the counting process like we did above using ordered samples, there is no constant that we can divide by in order to cancel out duplication. It turns out we need a whole new way to think about this situation if we are to come up with a fruitful method for counting the possibilities. Let's look at an easy example to illustrate these problems and get some idea about how to proceed.

✠ **EXAMPLE 4.4 - 6**

Consider a sample space S with three elements to draw from, say $S = \{a, b, c\}$. Determine how many samples of size two there are, if repetition is allowed.

Solution

The numbers involved in this problem are small enough so we can list all the samples. We'll write $<x, y>$ to indicate that x and y are the members of the sample. Order isn't important, so $<y, x>$ is the same sample as $<x, y>$, but we don't require that $x \neq y$.
Here is the list: $<a, a>$, $<a, b>$, $<a, c>$, $<b, b>$, $<b, c>$, $<c, c>$; 6 samples in all.
Now, we could have started out with all ordered pairs; that allows for repeated elements. There are $3^2 = 9$ ordered pairs here. However, there is no fixed duplication number to divide this total by. The ordered pairs (a, b) and (b, a), for example, count toward one sample $<a, b>$, so we should divide this part of the count by 2; but we can't divide by 2 in general, because (a, a) is present only once and has to be counted as one input.
We really need a gestalt shift to count these samples more easily. Instead of focusing on the two spots in the sample that we want to fill in with a letter, let's concentrate on the three letters that can be chosen. And, rather than put the letters into the sample, we'll distribute two winning tags among the letters a, b, and c to indicate how many times, if any, they've been chosen to be in the sample.
Using three separated blanks $\_\_ \mid \_\_ \mid \_\_$ to stand for the three letters in order, we need to count the number of ways we can assign two winning tags. If we use $*$ for our choice symbol, the sample $<a, b>$ would be represented by $\underline{*} \mid \underline{*} \mid \_\_$ and the sample $<c, c>$ by $\_\_ \mid \_\_ \mid \underline{*\,*}$. The blanks can obviously be ignored; the essential thing is the location of the stars relative to the separators. So we can represent these samples by $* \mid * \mid$ and $\mid \mid * *$.
The new question we want answered, therefore, is this: how many ordered 4-sequences of $|$'s and $*$'s are there that contain two stars? This is precisely the number of ways we can choose two positions in a 4-sequence for the two $*$'s: in $\binom{4}{2} = 6$ ways.

The following proposition and its proof generalizes the last example and gives us a formula. However, knowing and being able to *apply the method of the proof* is more important than memorizing the formula.

PROPOSITION* 4.4 - 2: *Counting Unordered Samples with Repetition

The total number of unordered samples of size k, allowing repetition, that can be drawn from a sample space of size n is $\binom{k + (n-1)}{k} = \dfrac{n \cdot (n+1) \cdots (n + (k-1))}{1 \cdot 2 \, \cdots \, k}$.

***Proof*:**

Create n compartments in some definite order to represent the n elements of the sample space, and separate them by $n - 1$ strokes.
Distribute k $*$'s to these n compartments in some way to represent choosing the elements. This generates sequences with $n - 1$ strokes and k $*$'s.

The number of different ways that this can be done is $\binom{k+(n-1)}{k}$. Expanding this expression and canceling the common terms $(n-1)!$ gives the final fraction. ■

Note the nice way in which the various ideas we've been studying come together in this proof. We wanted to count the number of unordered samples containing so many things, allowing repetitions. We did this by first forming ordered sequences to represent the process. Then, in order to count the sequences we were interested in, we counted combinations of so many positions within the sequence, without concern for the order in which they were chosen.

Note also that the fractional formulas for counting unordered samples, with or without repetition, have a rather satisfying symmetry about them. Both types of unordered samples have $k!$ in the denominator. For combinations without repetition, the numerator has k factors, starting with n and counting down. For combinations with repetition, the numerator also has k factors, but this time starting with n and counting up.

✠ EXAMPLE 4.4 - 7

A doting grandmother wants to give a total of 20 five-dollar bills to her four young grandchildren. In how many different ways can this dear lady distribute the money?

Solution

We can model this the way we counted unordered samples with repetition.
Think of four compartments, one for each child, each of which can receive any number of five-dollar bills from 0 to 20, until all 20 of them have been distributed.
This can be symbolized by a 23-sequence containing 3 compartment-separator symbols and 20 five-dollar symbols.
There are $\binom{23}{20} = \binom{23}{3} = \frac{23 \cdot 22 \cdot 21}{1 \cdot 2 \cdot 3} = 1771$ ways to do this.

Of course, if she doesn't want to be accused of favoritism, she'd better give them each $25.

✠ EXAMPLE 4.4 - 8

An *ordered partition of a finite number* N is a decomposition of the number N into p natural numbers $x_1, \ldots, x_p$ in order such that $x_1 + \cdots + x_p = N$.

a) In how many ways can $N = 4$ be decomposed into an ordered sum of two natural numbers? What if order isn't important?

b) In how many ways can a number N be decomposed into an ordered sum of p natural numbers? What if order isn't important?

Solution

a) We can decompose 4 into an ordered sum of two natural numbers in the following five ways: as $0+4$, $1+3$, $2+2$, $3+1$, and $4+0$.
More systematically, we can solve $x_1 + x_2 = 4$ for solution-pairs (x_1, x_2), using the following reasoning. Distribute four units (the quantity to be partitioned) to two compartments that represent the two variables.
Using the last proposition, there are $\binom{4+1}{4} = 5$ ways to do this.
If unordered sums are used, so that for instance $0+4$ and $4+0$ are essentially the same, there are only 3 partitions.

b) Now we generalize. To decompose N into a sum of p terms, think of distributing N units (say, $*$'s) into p compartments (one for each term) with $p-1$ separators.

This yields certain sequences of length $N + (p-1)$ ones in which N locations have been chosen for the units from among the total $N + (p-1)$ positions. There are $\binom{N+(p-1)}{N}$ of these.

If order of the summands isn't important, the problem is more difficult. I don't know of a simple formula for the number of unordered partitions of a number N into p summands, though there may be one. (Research project, anyone?)

EXERCISE SET 4.4

*1. *Creating a Counting Chart*
Create a 2×2 chart, giving the counting formulas for all of the various situations we've encountered: ordered vs. unordered, repetition allowed vs. repetition not allowed.

Problems 2 - 10: Counting Events and Possibilities

Use the methods of this section to work the following exercises.

2. *Diagonals in a Polygon*
   a. How many diagonals can be drawn in a convex pentagon (a 5-sided figure with no indentations)?
   b. How many diagonals can be drawn in a convex polygon of n sides?

*3. *Polite Handshakes*
   *a. At a party of 15 people, everyone shook hands with everyone else. How many handshakes took place?
   b. Find a formula for the number of handshakes if n people were present at the party and each person shook hands with everyone else.

*4. A *Discrete Mathematics* class of 18 students has 7 women in it. If 3 students are picked each period to exhibit their homework solutions on the board, how many different groups are possible that contain:
   *a. Any number of women, from 0 to 3.
   *b. Exactly 1 woman.
   c. 3 women.
   *d. No more than 1 woman.

*5. *Paths in a Grid*
   *a. How many different paths can be drawn along an integer grid to go from the origin $(0, 0)$ to the point $(2, 3)$ if one is only allowed to go either up or right at each integer lattice point? How many paths are there from the origin to the point (m, n)? Explain.
   b. How many different ways are there to pass from the triple $(0, 0, 0)$ to the point $(3, 4, 5)$ if you can only increase one coordinate at a time by adding 1 to it? Explain.

*6. Sara makes a donut-run for her staff every morning. She always buys a dozen donuts, choosing from five different types of donuts.
   *a. If there are no restrictions on what she should buy, how many different donut-dozens are possible?
   *b. If she always gets at least one of each type of donut, how many different donut-dozens are possible?

EC 7. *Depleting the Piggy Bank*
   a. If 5 coins are taken from a piggy bank containing many pennies, nickels, dimes, and quarters, how many different collections of (types of) coins are possible?
   b. How many different total amounts of money (the total value of the coins) are possible? Explain.

8. An office mailroom has 15 mailboxes for its employees. In how many different ways can 22 pieces of mail be distributed to these mailboxes? Explain.

*9. A committee of 5 members is seated in a circle around a table.
   *a. How many essentially different arrangements of the members are possible if there are 5 chairs? if there are 6 chairs and 1 remains empty? if there are 7 chairs and 2 remain empty?
   *b. Calculate the number of essentially different arrangements for part *a* if relative location among the members and empty places (rather than which chair each occupies) is all that counts.
   c. If the orientation of the circle is unimportant, what would your answer to part *b* be?

*10. In how many distinct arrangements can you put the letters from the word MISSISSIPPI? Explain your reasoning.

Problems 11 - 12: True or False

Are the following statements true or false? Explain your answer.

*11. For all n and k, $C(n,k) \le P(n,k)$.

12. For all $k \le n$, $\binom{n}{n-k} = \binom{n}{k}$.

Problems 13 - 14: Counting for Games

The following problems ask you to count pieces in games.

*13. *Dominoes*
A domino from a set of *Double-Twelve Dominoes* is a flat rectangular piece whose face is divided in half, each part containing between 0 and 12 dots on it in a recognizable pattern. If there is exactly one domino for each possible pair of numbers, how many dominoes are there in a full set? Explain.

EC 14. A *tri-omino* is a flat triangular piece having a number from 0 through 5 placed in each corner.
   a. If a set of tri-ominoes has all possible configurations of the numbers, oriented clockwise in order of increasing (non-decreasing) size, how many pieces are needed for a full set?
   b. How many pieces would be needed if tri-ominoes were oriented in any way, clockwise or counter-clockwise?

Problems 15 - 17: Counting and Probability

The following problems involve counting and discrete probability.

*15. How many 5-card hands from a standard deck of 52 cards contain the following.
   a. One pair (two cards of the same kind plus three cards of different kinds from the pair and from one another).
   *b. Two pairs (two of one kind plus two of another kind plus one card of a third kind).
   *c. A straight (a run of five consecutive values in any suits, where the non-numerical face cards are ordered after the ten as jack, queen, king, and ace).
   *d. Determine the probabilities associated with each of the above hands and compare them to one another and the probabilities calculated in Example 5.

16. An urn contains 5 red balls, 8 white balls, and 10 blue ones.
   a. How many different sets of 3 red, 3 white, and 3 blue balls can be taken out of the urn?
   b. What is the probability of drawing 3 of each color if 9 balls are drawn from the urn?

17. Yahtzee is a game played by two or more players each rolling 5 dice in turn, with two chances to re-roll some or all of the dice in order to try to get a score in 1 of 13 different categories.
   a. What it is the probability of rolling a full house (three of one kind, two of another) in one toss?
   b. What is the probability of rolling a small straight (four numbers in a row) on one toss?
   c. What is the probability of rolling a large straight (five numbers in a row) on one toss?
   d. What is the probability of rolling a Yahtzee (all five numbers the same) on one toss?
   e. If a player rolls three 5's, a 2, and a 3, what is her probability of getting a Yahtzee on her next two rolls if she decides to keep the 5's and roll the other two dice again?

Problems 18 - 25: Pascal's Triangle

The following problems explore some of the many patterns that have been discovered in Pascal's Triangle.

*18. Using the factorial formula for the binomial coefficients involved, prove the basic recursion formula on which Pascal's triangle depends: $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$. Explain why this formula is basic to generating the triangle of coefficients.

EC 19. Prove that $\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$, this time using the fact that $\binom{n}{k}$ is the number of subsets of size k in a set S of size n. *Hint*: pick some element of S and partition the collection of subsets into two classes, depending on whether the element is in the subset or not.

20. Prove the *Binomial Expansion Theorem* rigorously using mathematical induction. Make use of Problem 18 where appropriate.

*21. Example 2 establishes that $\sum_{k=0}^{n} \binom{n}{k} = 2^n$. What is $\sum_{k=0}^{n} (-1)^k \binom{n}{k}$? Prove your result using a binomial expansion.

22. Show that $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$ when $n = 3$ by direct calculation. Then show via the recursive formula in Problem 18 how this formula comes about: trace the value of $\binom{6}{3}$ back up Pascal's triangle until you reach the sides of the triangle (row $n = 3$).

EC 23. Prove the general result stated in Problem 22.

24. Identify the pattern of the following result and check it using Pascal's Triangle when $n = 6$. Then prove in general that $\sum_{k=1}^{n} \binom{k}{1} = \binom{n+1}{2}$.

25. Patterns like the one in Problem 24 exist elsewhere in Pascal's Triangle. Find another one, formulate it as a proposition, and prove it.

HINTS TO STARRED EXERCISES 4.4

1. The formulas needed are all in the text.

3. a. Each handshake involves 2 people out of the group of 15.

4. a. This can be worked in more than one way. The simplest is to note that every exhibition group will have between 0 and 3 women in it.
   b. An exhibition group with 1 woman will also contain 2 men.
   d. Calculate two separate values (no women, one woman) and combine them.

5. a. How many times must you go right (R), and how many times must you go up (U)? Put R's and U's together in a map instruction string and count the possibilities.

6. a. Make donut compartments and indicate your 12 choices with a $*$ or a 0.
   b. Use the same model as in part a, only take care of the requirement before counting.

9. a. This is a permutation problem.
   b. Divide your answers in part a by the duplication possible.

10. Start with a permutation and then divide out by the duplication introduced by the letters I, S, and P.

11. [No hint.]

13. You need to choose 2 numbers from $0-12$, allowing repetition but ignoring order.

15. See Example 5.

18. This is straight-forward algebra of fractions involving factorials.

21. Relate this to an expansion of $(a+b)^n$ for $b=-1$ and an appropriate value of a.

4.5 Additive Counting Principles

The part of combinatorics we have focused on so far involve counting principles that are built upon the basic *Multiplicative Counting Principle*. These principles derive ultimately from *Proposition* 4.3.1: *the cardinality of a Cartesian product is the product of its cardinalities.*

This section will look at an *additive* counting situation and its set-theoretic background. In its most basic form, we will count the number of elements in the union of two sets; here the *Additive Counting Principle* will provide the necessary foundation. This is a simpler counting situation than what we looked at earlier, but we had to wait until now to present it because its most general formulation requires a knowledge of how to count combinations.

Cardinality of Finite Sets and Unions

The number of elements in two finite sets taken together can be found by counting. If the sets overlap, the common elements should be counted only once. Otherwise, the total can be found by adding the two cardinalities: simple addition handles exclusive alternatives.

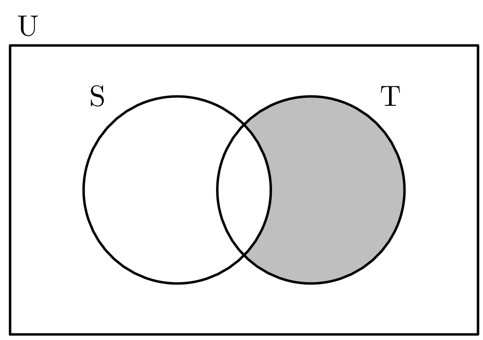
In a rigorous development of *Set Theory*, the following principle would be taken as the general definition for adding cardinal numbers. Here we will postulate it as an axiom stipulating how the cardinality of a disjoint union of finite sets is determined. The *Additive Counting Principle* follows immediately out of this.

AXIOM* 4.5 - 1: *Cardinality of Disjoint Unions
If S and T are disjoint finite sets, $|S \cup T| = |S| + |T|$.

PROPOSITION* 4.5 - 1: *Additive Counting Principle
If an outcome can occur in one of two mutually exclusive ways, the first in m ways and the second in n ways, then the number of ways the outcome can occur in either way is $m + n$.

Now, what if the sets are not disjoint but overlap? Then adding cardinalities counts the common part twice, so it must be subtracted once to compensate. This method of counting is justified by applying the above results.

PROPOSITION* 4.5 - 2: *Cardinality of Unions
$|S \cup T| = |S| + |T| - |S \cap T|$.



***Proof*:**
We'll break $S \cup T$ into two disjoint parts and use that to relate their cardinalities.

$$S \cup T = S \cup (T - S)$$

Therefore, since these sets are disjoint,

$$|S \cup T| = |S \cup (T - S)| = |S| + |T - S| .$$

Similarly, we have $|T| = |T - S| + |S \cap T|$.
Thus, $|T - S| = |T| - |S \cap T|$.
Substituting this value in the equation above, we have $|S \cup T| = |S| + |T| - |S \cap T|$. ∎

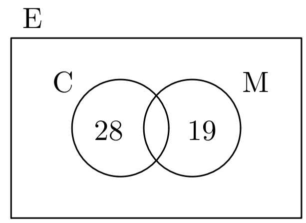
COROLLARY*: *Generalized Additive Counting Principle
If an outcome can occur in one of two ways, the first in m ways and the second in n ways, then the number of ways it can occur in either way is $m + n - b$, where b is the number of outcomes that occur in both ways.

✠ **EXAMPLE 4.5 - 1**

The department secretary sent out an email to all mathematics and computer science majors. There are 19 mathematics majors and 28 computer science majors, yet the email only went out to 43 students in all. Explain how this happened.

Solution

Let C represent those students having a computer science major and M those majoring in mathematics.
Then $|C| + |M| = 28 + 19 = 47$.
Since $|C \cup M| = 43$, $|C \cap M| = 47 - 43 = 4$.
Thus, four students must be double-majoring in mathematics and computer science.



Principle of Inclusion and Exclusion for Three Sets

Working with two sets isn't difficult and doesn't really need any fancy principles: you just draw a diagram and use your common sense for doing some simple arithmetic. But when more sets are involved, things begin to get complicated. We'll first focus on the situation where there are three sets; this case can still be illustrated and worked using a standard Venn diagram.

If the sets are pairwise disjoint so that no two of them overlap, it's easy to count their union: take the individual cardinalities and add them up. This is a straight-forward extension of the *Additive Counting Principle*, and it applies to any number of sets. In fact, this extension is more or less the basis for how we will treat the more complex situation; breaking the sets up into pairwise disjoint components we can determine the cardinality of any one of them by knowing that of the others.

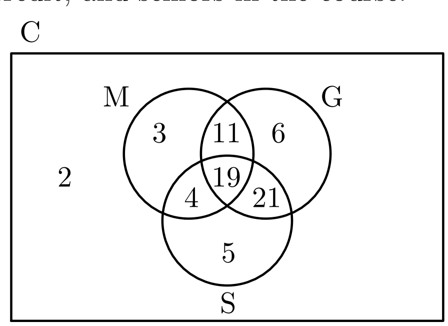
✠ **EXAMPLE 4.5 - 2**

An upper-level general education (GE) class has 71 students enrolled. Of these students, 37 are men, 57 are taking it for GE credit and the rest are taking it as an elective, 49 are seniors, 7 men are taking it as elective, 23 men are seniors, 40 seniors are taking it for GE credit, and 21 senior women are taking it for GE credit. How many non-senior women are taking it as an elective?

Solution

The diagram below shows the final result for the class C; it would be good to take a blank diagram and fill in the numbers step by step. The sets M, G, and S represent respectively men in the course, students taking the course for GE credit, and seniors in the course.

Start with the fact that 21 senior women are taking the course for GE credit: that enumerates a single inner component. Since 40 seniors are taking it for GE credit, 19 must be men. This means 4 senior men are taking it as an elective and 3 non-seniors. Therefore, 11 non-senior men are taking it for GE credit. This leaves 6 non-senior women taking it for GE credit. Since 49 seniors are in the class, there must be 5 senior women taking it as an elective. In a class of 71 students, that leaves only 2 non-senior women taking it as an elective.



Now consider any three finite sets S_1, S_2, and S_3. How is the cardinality of the union $S_1 \cup S_2 \cup S_3$ related to the various parts? If we add up the members in each set, we've then counted the members in the intersections $S_1 \cap S_2$, $S_1 \cap S_3$, and $S_2 \cap S_3$ twice, so we should subtract those numbers once from the total. But wait: we've actually counted the members of

$S_1 \cap S_2 \cap S_3$ three times, once for each set. So if we subtract off the numbers for each double intersection, we will now have subtracted off the number in the triple intersection three times, so it won't have been counted at all. Well, then, we'd better add that back in. In symbols we have:

$$|S_1 \cup S_2 \cup S_3| = (|S_1| + |S_2| + |S_3|) - (|S_1 \cap S_2| + |S_1 \cap S_3| + |S_2 \cap S_3|) + |S_1 \cap S_2 \cap S_3|$$

In words: the cardinality of a triple union is the sum of the cardinalities of the individual sets minus the sum of the cardinalities of the double intersections plus the cardinality of the triple intersection.

This is a special case (for three sets) of the more general *Principle of Inclusion and Exclusion*, which we'll formulate and prove shortly. But first let's take another look at the last example.

✠ **EXAMPLE 4.5 - 3**

Rework Example 2, using the formula just developed. Comment on its efficiency.

Solution

The quantity we're interested in is $|\overline{M \cup G \cup S}|$.

Since $(M \cup G \cup S) \subseteq C$, $|\overline{M \cup G \cup S}| = |C| - |M \cup G \cup S|$. [See Exercise 11]

By the above formula,

$$\begin{aligned} |M \cup G \cup S| &= (|M| + |G| + |S|) - (|M \cap G| + |M \cap S| + |G \cap S|) + |M \cap G \cap S| \\ &= (37 + 57 + 49) - (|M \cap G| + 23 + 40) + |M \cap G \cap S| \end{aligned}$$

Since $7 = |M - (M \cap G)| = |M| - |M \cap G| = 37 - |M \cap G|$, we have $|M \cap G| = 30$.

And since $|G \cap S - M \cap G \cap S| = 21$, $|M \cap G \cap S| = 19$.

Substituting these two new values in the above equation, we get

$$\begin{aligned} |M \cup G \cup S| &= (37 + 57 + 49) - (30 + 23 + 40) + 19 \\ &= 143 - 93 + 19 \\ &= 69 \end{aligned}$$

Thus, $|\overline{M \cup G \cup S}| = 71 - 69 = 2$.

It's not really any faster to use the given formula for three sets unless the sets whose cardinalities we know are the ones appearing in the formula. Otherwise cardinalities need to be determined using other relations that may still best be figured out by looking at the diagram, as we did in Example 2. However, the formula does give us a systematic way to work these sorts of problems, one that can be generalized to situations where we can no longer diagram very easily what's going on.

Principle of Inclusion and Exclusion in General

Suppose we have any number of sets and we want to count the total number of elements in their union. How can we do this in terms of its component subsets? If we have pairwise disjoint components, the total is found by simple addition of the individual cardinalities. But if we don't, we have somehow to account for the duplication involved, subtracting and adding various values to arrive at the correct count. How to do this in terms of all possible intersections is the content of the *Principle of Inclusion and Exclusion*. It generalizes the method we just illustrated for three sets. This result is best stated largely in words since a fully symbolic formulation would be messy.

THEOREM* 4.5 - 1: *Principle of Inclusion and Exclusion

Let $S_1, S_2, \ldots, S_n$ be a collection of n sets. $|S_1 \cup S_2 \cup \cdots \cup S_n|$ is the sum of the cardinalities of all possible odd-fold intersections $S_{k_1} \cap \cdots \cap S_{k_o}$ minus the sum of the cardinalities of all possible even-fold intersections $S_{k_1} \cap \cdots \cap S_{k_e}$.

***Proof*:**

Our proof strategy is to show that in the end the given formula counts each element of the total union *exactly once.*

To do this, we begin with a simple but potent observation: each element of the total union belongs to exactly m sets of the collection (namely, all the sets to which it belongs) for some unique m, $1 \le m \le n$.

So, let x be an arbitrary element of the union, and let m denote the number of sets in the collection to which it belongs.

As a warm-up to determining how many times the formula counts x, note that in adding up the cardinalities for all the individual sets (the 1-fold intersections), the formula counts x a total of $\binom{m}{1} = m$ times, once for each set it's in.

Let's do one more situation and then generalize. In subtracting off the cardinalities for all two-fold intersections in the formula, we will have counted x (negatively) a total of $\binom{m}{2}$ times since this is the number of two-fold intersections x belongs to.

In general, then, for each k, $1 \le k \le m$, the formula counts x a total of $\binom{m}{k}$ times since this is the number of k-fold intersections x is in — either adding it in (for odd k) or subtracting it off (for even k). The parts of the formula that involve intersections of more than m sets are irrelevant to x; they won't count x at all.

Thus x gets counted a total of $\binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{k+1}\binom{m}{k} + \cdots + (-1)^{m+1}\binom{m}{m}$ times. We need to evaluate this *alternating series* to show that this is 1, as it should be.

This is best seen by cleverly expanding $(1 + {}^{-}1)^m$ using the *Binomial Expansion Theorem* (see Exercise 4.4-21):

$$0 = (1 + {}^{-}1)^m = \sum_{k=0}^{m} \binom{m}{k} 1^{m-k}(-1)^k = \binom{m}{0} - \binom{m}{1} + \cdots + (-1)^m \binom{m}{m}.$$

Solving this for our desired sum, we have $\binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{m+1}\binom{m}{m} = \binom{m}{0} = 1.$

Thus our formula counts each x in the union exactly once, just as it should. ■

***COROLLARY*: *Principle of Inclusion and Exclusion* (*Complementary Form*)**

Let $S_1, S_2, \ldots, S_n$ be a collection of n subsets of a set S. Then $|\overline{S_1 \cup S_2 \cup \cdots \cup S_n}| = |S| - |S_1 \cup S_2 \cup \cdots \cup S_n|$, which is $|S|$ plus the sum of the cardinalities of all possible even-fold intersections $S_{k_1} \cap \cdots \cap S_{k_e}$ minus the sum of the cardinalities of all possible odd-fold intersections $S_{k_1} \cap \cdots \cap S_{k_o}$.

***Proof*:**

This follows immediately from the rule for calculating cardinalities of complements (see Exercise 11) and the *Principle of Inclusion and Exclusion.* ■

✠ **EXAMPLE 4.5 - 4**

How many positive integers less than or equal to 360 are relatively prime to 360 (i.e., have no factors besides 1 in common with 360)?

Solution

We'll first factor: $360 = 2^3 3^2 5$. So we must find the number of positive integers that have no factor of 2, 3, or 5.

Let S_2 be the set of all multiples of 2 less than or equal to 360,
S_3 be the set of all multiples of 3 less than or equal to 360,
S_5 be the set of all multiples of 5 less than or equal to 360,
and let $S_1 = \{n : n \leq 360\}$.

We must find $|S_1 - (S_2 \cup S_3 \cup S_5)| = |S_1| - |(S_2 \cup S_3 \cup S_5)|$.

$|S_1| = 360$, $|S_2| = 360/2 = 180$, $|S_3| = 360/3 = 120$, and $|S_5| = 360/5 = 72$.

Also, $|S_2 \cap S_3| = 360/6 = 60$, $|S_2 \cap S_5| = 360/10 = 36$, and $|S_3 \cap S_5| = 360/15 = 24$.

Finally, $|(S_2 \cap S_3 \cap S_5)| = 360/30 = 12$.

Using the *Principle of Inclusion and Exclusion*,

$$\begin{aligned}|(S_2 \cup S_3 \cup S_5)| &= (|S_2|+|S_3|+|S_5|) - (|S_2 \cap S_3|+|S_2 \cap S_5|+|S_3 \cap S_5|) + |(S_2 \cap S_3 \cap S_5)| \\ &= (180 + 120 + 72) - (60 + 36 + 24) + 12 \\ &= 372 - 120 + 12 = 264\end{aligned}$$

Thus, there are $360 - 264 = 96$ numbers less than or equal to and relatively prime to 360.

EXERCISE SET 4.5

Problems 1 - 8: Counting Sets

Work the following problems, using either a Venn diagram or the Principle of Inclusion and Exclusion to assist you as needed.

*1. *Card Hands*
   *a. How many five-card hands contain three aces? two kings? three aces and two kings? three aces or two kings?
   b. How many five-card hands are three of a kind? two of a kind? a full house (three of one kind plus two of another)? three of a kind or two of a kind?
   c. How many five-card hands are a flush (five non-consecutive cards in the same suit)? How many hands are a straight flush (five consecutive cards in the same suit, Ace being either high or low)? How many hands are a straight (five consecutive cards in any suits)? How many hands are a flush, a straight flush, or a straight?

*2. There are 150 faculty at a technical community college: 100 faculty are full-time; 60 faculty are women, but only 25 of these are full-time; 40 faculty teach a liberal arts course, of which 30 are women and 20 are full-time; and 10 full-time women faculty teach a liberal arts course. How many full-time men faculty do not teach a liberal arts course? How many part-time men faculty does the college employ? Explain your reasoning.

3. A bowl of fruit contains 4 bananas, 5 apples, and 6 oranges.
   a. In how many different ways can three pieces of fruit be chosen so that all three fruit are the same?
   b. In how many different ways can three pieces of fruit be chosen so that at least two fruits are the same?

4. A one-room school house contains 20 children. Of these students, 14 have brown eyes, 15 have dark hair, 17 weigh more than 80 pounds, and 18 are over four feet tall. Show that at least 4 children have all four features.

*5. *Counting Primes*
   Using the *Principle of Inclusion and Exclusion*, *Complementary Form*, determine how many primes there are in the first 100 integers.
   Hint: Every number is either prime or the multiple of a prime number that is less than or equal to the number's square root (why?).

*6. *Binary Events*

*a. How many different bytes (eight-bit strings; see Example 4.3-3) do not contain 5 consecutive 1's?

*b. A coin is flipped 8 times. What is the probability of having the coin land heads up 4 or less times in a row? Assume that heads and tails are equally likely to occur for each toss.

*7. A loose-change jar contains 62 coins: 31 pennies, 10 nickels, 12 dimes, and 9 quarters. In answering the following questions, do not distinguish between collections that have the same number of coins of the same type and assume that each coin is as likely as the next to be chosen.

*a. How many different collections of 7 coins can be drawn out of this jar?

b. How many different collections of 7 coins can be drawn out having at least one coin of each type? What is the probability of withdrawing such a collection?

*c. How many different collections of 7 coins have 2 quarters or 2 dimes? 2 quarters, 2 dimes, or 2 nickels?

EC *d. What are the probabilities associated with choosing the collections in part c?

8. *Illustrating the Principle of Inclusion and Exclusion*

a. Write out the full formula for $|S_1 \cup S_2 \cup S_3 \cup S_4|$.

b. Write out the full formula for $|S_1 \cup S_2 \cup S_3 \cup S_4 \cup S_5|$ if all 4-fold intersections are empty.

Problems 9 - 10: True or False

Are the following statements true or false? Explain your answer.

9. The cardinality of the union of two sets is the sum of the set's cardinalities.

*10. If a first action can be performed in m ways and for each of these a second different action can be performed in n ways, the joint action can be performed in $m + n$ ways.

Problems 11 - 17: Cardinality and Subsets

Suppose all the sets below are finite sets. Prove the following without using the Principle of Inclusion and Exclusion or its Corollary.

*11. $S \subseteq U \rightarrow |\overline{S}| = |U - S| = |U| - |S|$ [This can be used as the definition of subtraction in a development of set-theoretic arithmetic.]

*12. Find and prove a general formula for $|S - T|$, regardless of how S and T are related.

13. $|S \cup T| = |S - T| + |S \cap T| + |T - S|$

14. $S \subseteq U \rightarrow |S| \leq |U|$ [Cardinality is monotone increasing.]

15. $|S \cap T| \leq |S| \leq |S \cup T|$

16. For a finite collection of sets, which number is smaller: the sum of the cardinalities of each set, or the cardinality of its union? Formulate a precise answer symbolically and then prove it.

*17. Suppose $\{S_i\}_{i=1}^{n}$ forms a partition of a set S. Prove by induction that $|S| = \sum_{i=1}^{n} |S_i|$.

Problems 18 - 31: Exploring Euler's Phi-Function

Use the Principle of Inclusion and Exclusion to help answer the following questions about Euler's ϕ-function, where $\phi(n)$ equals the number of positive integers less than or equal to n that are relatively prime to n (have no factors besides 1 in common with n).

*18. Determine $\phi(21)$, the number of positive integers relatively prime to 21. How is $\phi(21)$ related to the factors of 21?

19. Determine a formula for $\phi(pq)$, where p and q are distinct primes. Put your formula in factored form, and prove your result. Does your formula hold if p and q are the same prime number? Does it hold if the numbers themselves are relatively prime?

*20. Determine $\phi(25)$. Relate this value to the prime factor of 25. Find a formula for $\phi(p^2)$, where p is a prime. Prove your result.

*21. Determine $\phi(105)$. Relate this number to the prime factors of 105.

EC 22. Determine a formula for $\phi(pqr)$, where p, q, and r are distinct primes. How is this expression related to the factors of pqr? Does your formula hold if the number is p^3 instead of pqr? If it is p^2q? If the numbers p, q, and r are relatively prime to one another? Justify your answers.

23. Determine $\phi(27)$. Relate this value to the prime factor of 27. Find a formula for $\phi(p^3)$, where p is a prime. Prove your result.

24. Determine $\phi(546)$ using the *Principle of Inclusion and Exclusion.* How is this value related to the prime factors of 546?

25. State and prove a formula for $\phi(pqrs)$, where p, q, r, and s are distinct primes. Use the *Principle of Inclusion and Exclusion.*

EC 26. Generalize the results of Problems 19, 22, and 25 to obtain a formula for $\phi(p_1 \cdots p_n)$ for the product of n distinct prime numbers p_i.

EC 27. Determine and prove a formula for $\phi(p^k)$, where p a prime number and k a positive integer.

28. Use the *Principle of Inclusion and Exclusion* to develop a formula for $\phi(p^m q^n)$ for distinct primes p and q. Take $m = 2$ and $n = 3$ for a case study if this helps. Prove that $\phi(p^m q^n) = \phi(p^m)\phi(q^n) = p^{m-1}q^{n-1}(p-1)(q-1)$. Does this formula still work if $m = 1 = n$?

EC 29. Generalize the results of Problems 26 – 28 to develop a formula for $\phi(m)$ for any positive integer m whose prime factorization is $m = p_1^{k_1} \cdots p_n^{k_n}$.

EC 30. Given your formula from Problem 29, show that $\phi(m) = m \cdot \prod_{p|m} (1 - 1/p)$, where p is any prime divisor of m.

31. Use your work from Problems 28 and 29 to show that $\phi(ab) = \phi(a)\phi(b)$ if a and b are relatively prime.

32. The Babylonians had a *sexagesimal* (a base sixty place-value) numeration system. This has some definite advantages for expressing fractions as a whole number of a smaller unit (sixtieths, thirty-six hundredths, etc); think of minutes and seconds of an hour or an angle, which are based on this system.
    a. How many fractions $1/n$ with $n \leq 3600$ can be so expressed?
    Recall: $1/n$ can be expressed as some fraction $k/3600$ iff n's prime factors are all among 3600's prime factors (why?). Thus, $1/8$ can be so expressed (as $450/3600$), while $1/14$ can't because 14 has a prime factor (7) that 3600 doesn't.
    b. How many proper fractions m/n, $m < n$ can be so expressed? (For this part, count $1/2$ as different from $30/60$.)
    c. How many distinct proper fractional values m/n can be so expressed (i.e., m/n is in reduced form; now count $1/2$ as the same as $30/60$)?

HINTS TO STARRED EXERCISES 4.5

1. a. To find the number of hands with 3 aces or 2 kings, treat it as the cardinality $|A_3 \cup K_2|$.

2. See Example 2.

5. For each prime less than or equal to $\sqrt{100}$, find the number of its *composite* (non-prime) multiples less than or equal to 100. Then use the *Principle of Inclusion and Exclusion* to find the total number of composite numbers less than or equal to 100. Keep in mind that 1 is not prime.

6. a. One approach you can use is to find the number of bytes with exactly 8 consecutive 1s, with exactly 7 consecutive 1s, exactly 6, and exactly 5. Then subtract this amount from the total number of bytes.
   b. Use your work from part *a*.

7. a. Since there are at least 7 of each type of coin, this problem can be solved by making four compartments for the different coin types and assigning 7 $*$'s to the compartments.
   c. To count the possibilities, use the method of part *a* along with the *Principle of Inclusion and Exclusion* to determine how many different collections have the required properties.
   d. To determine the probabilities of the collections, you can't just take the calculated number of collections of that type and divide by the total number of collection types, since some collections are more probable than others (7-penny collections are much more likely than 7-quarter collections, for instance); here you need to distinguish individual coins to account for their frequency. Thus, for example, there are $\binom{9}{2} \cdot \binom{60}{5}$ different collections of 7 coins having exactly 2 quarters in them out of a total of $\binom{62}{7}$ equally likely 7-coin collections.

10. [No hint.]

11. Start by letting $|U| = u$ and $|S| = s$. You'll need to use Axiom 1 in your proof.

12. Draw a Venn diagram to help you find the formula to prove. In your proof, note that $S - T$ and $S \cap T$ form a partition of S (explain why in your proof). You'll also need to use Axiom 1.

17. Prove this by induction, using Axiom 1 where it is relevant.

18. $\phi(21) = 12$. When figuring out the relation between $\phi(21)$ and the factors of 21, make use of Proposition 2.

20. The formula you need to prove is $\phi(p^2) = p^2 - p$.

21. If the prime factorization of a number is pqr, then there are qr multiples of p, pr multiples of q, and pq multiples of r less than or equal to pqr. Keeping this fact in mind, make use of the *Principle of Inclusion and Exclusion*.