Discrete Mathematics: Chapter 1, Sentential Logic

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Discrete Mathematics: Chapter 1, Sentential Logic

Abstract
As a general field of study, logic isn't really a branch of mathematics. It deals with consequential reasoning, something we do in all areas of our lives. It enters into daily conversation (“How can you believe that? Don’t you know that . . .?”) and cooking decisions (“To modify this recipe to feed four instead of six, I need to . . .”) as well as academic studies (“If poverty is a factor in systemic educational failure, then we should . . .”). Any time we draw a necessary conclusion from something we already know, logical processes come into play.

So mathematics and computer science cannot lay exclusive claim to logic. Nonetheless, deductive reasoning is a crucial ingredient in all quantitative sciences. Moreover, mathematics and logic have had an ongoing intimate relationship since ancient times, and computer science has been closely allied with logic since it began in earnest in the twentieth century. In fact, the connections have become so close over the last century that portions of logic are difficult to distinguish from mathematics, and computer science sometimes seems like applied logic. So it makes sense for us to spend some time here with logic, regardless of how it is classified.

In this first section, we'll begin putting this mathematics-logic-computer science nexus into historical perspective, and then we’ll discuss what sorts of things we can expect from logic for our study of discrete mathematics. In the process we’ll introduce some key notions that will be developed in more detail later in the book.

Keywords
axiomatic set theory, logic, proof, algebra, functions

Disciplines
Christianity | Computer Sciences | Mathematics

Comments
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Chapter 1

SENTENTIAL LOGIC
1.1 Introduction to Logic and Proof

As a general field of study, logic isn’t really a branch of mathematics. It deals with consequential reasoning, something we do in all areas of our lives. It enters into daily conversation (“How can you believe that? Don’t you know that . . .?”) and cooking decisions (“To modify this recipe to feed four instead of six, I need to . . .”) as well as academic studies (“If poverty is a factor in systemic educational failure, then we should . . .”). Any time we draw a necessary conclusion from something we already know, logical processes come into play.

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In this first section, we’ll begin putting this mathematics-logic-computer science nexus into historical perspective, and then we’ll discuss what sorts of things we can expect from logic for our study of discrete mathematics. In the process we’ll introduce some key notions that will be developed in more detail later in the book.

Mathematics and Logic in Historical Perspective

Mathematics seems the very model of systematic science. Things are done in an orderly step-by-step fashion, one thing building on another. All the parts are interconnected by a consistent network of ideas and procedures. Each statement is precisely worded, and a careful train of reasoning proves it to be true. We owe this picture of mathematics and the associated deductive methodology to the ancient Greeks, who first organized mathematics into an axiomatic system about 2500 years ago.

Theory and reasoning were all-important to the Greeks. Unlike their predecessors, the Greeks treated geometry as a theoretical science, not as a collection of procedures for calculating mundane measurement results. And it was number theory, not arithmetic, that received their loving attention, because it led to universal knowledge, such as the proposition that every number has a prime factor. Numerical computations, on the other hand, were performed following patterns exhibited in particular problems. They were best left to those immersed in the workaday world of commercial trade and government bureaucracy.

In the beginning, then, deductive reasoning was closely associated with theoretical mathematics but not computation. Mathematical theories began with a set of basic assertions, called postulates or axioms, and with definitions that told exactly how terms were to be used. They then proceeded through a series of theorems, each result being strictly demonstrated on the basis of previously proved propositions, postulates, or definitions. Numerical illustrations and practical applications were notoriously absent from classical Greek mathematics.

Valid reasoning occurred elsewhere in Greek culture, too, such as in philosophy and public debates, but the successful development of axiomatized mathematical theories was a strong positive impetus behind the rise of logic. The first system of deductive logic was created around 325 B.C. by the philosopher Aristotle, whose analysis of deductive argument was accepted as nearly the final word on the subject for over 2000 years. Aristotle also put forward a foundationalist theory of knowledge modeled on mathematics: true science begins with self-evident truths intuited about some aspect of the world and uses them as an axiomatic basis on which to build the entire science via deductive reasoning. Logic
and mathematics joined forces to model how natural knowledge should be pursued and organized. Ironically enough, however, mathematics’ own internal proof-structures did not adhere to the specific syllogistic forms of reasoning Aristotle had identified as basic.

In the early modern period, important advances were made in computational arithmetic (adopting our base-ten positional system; introducing logarithms) and in algebra (solving third- and fourth-degree equations; using symbolic reasoning), and experimentation and inductive reasoning were added as crucial components of natural science. Along with these developments came a decreased emphasis on logic and deductive reasoning, even in mathematics. Topics that lacked a sound theoretical basis became the principal research interest of mathematicians. Algebra was combined with geometry to create coordinate geometry, and this was further extended to include calculations with infinitesimals and infinitely many numbers, giving birth to calculus and differential equations. New and powerful mathematical procedures such as differentiation and infinite series representations allowed mathematicians to discover new results in mathematics and physics, but they also outstripped mathematicians’ ability to justify them deductively. Some of the most basic ideas then in use were poorly understood and explained.

In the nineteenth century, both for educational and mathematical reasons, mathematicians began to pay closer attention to the theoretical foundations of calculus. They also investigated the logical basis of number systems and calculation procedures and thought more deeply about the real meaning and justification of algebra.

About the same time, some philosophers were beginning to press for a revival of deductive logic. This did not lead to a mere reaffirmation of Aristotelian logic, however. By the middle of the century, mathematicians such as Boole and De Morgan had turned to logic and had suggested some extensions of the field and some new approaches. De Morgan proposed that logic should be expanded to treat relations, something mathematics has in abundance. In the case of Boole an unusual version of algebra was used to formulate logical statements and to derive conclusions from premises. This was quite a maverick approach, for at this time even ordinary algebra had no adequate theoretical basis. To use the area of mathematics farthest removed from deductive rigor (computational algebra) as the intellectual instrument for developing the general science of deductive inference (logic) seemed extremely wrong-headed. But Boole’s work contained the seeds of the close coordination between mathematics, logic, and computation that would eventually give birth to computer science in the twentieth century.

Before that happened, however, logic was radically transformed by another group of mathematicians. Frege, and later Russell, thought that logic should be made the foundation for mathematics, starting with arithmetic, instead of the other way around. So they tipped Boole’s program on its head. Ultimately this logicist program failed to accomplish its aim of reducing mathematics to logic, but in the process of trying, an important reform of logic was undertaken. The result was a new system of logic that was nicely fitted to mathematics. This is the system we now use and which goes under various names, such as Symbolic Logic or Mathematical Logic.

Variants of this system exist, differing especially with respect to how deduction is treated. The earliest systems were formulated with the primary aim of providing a logical foundation for mathematics. They were more concerned with symbolizing and encapsulating the results of mathematics within logic than with capturing its method of reasoning. In the mid-1930s, however, some mathematicians began to explore a natural deduction system, developing logic to handle the way people actually deduce conclusions from premises. This approach is better suited for analyzing and constructing mathematical proofs, one of the main reasons we have for studying logic here, so we will follow their lead. Our approach derives from the work of Jaskowski, a Polish mathematician, as subsequently modified by Fitch, an American logician.

Over the last century and a half, then, logic has become increasingly allied with (some think, taken over by) mathematics, not only because of its perceived significance for the the-
oretical foundations of mathematics, but also for its analysis of mathematical reasoning. Not unconnected with these developments, logic has also become more closely associated with algebra and computation. Boole’s work was the first successful step in this direction, though others before him had anticipated such a development. Here an algebra of 0 and 1 was closely connected with an elementary part of logic. In the late 1930s Shannon discovered that Boolean algebra could be used to analyze and design switching circuits for doing numerical computations and other information processing. Soon people were building circuits with logic gates representing Boolean operators such as not, and, and or and using them to do multi-purpose calculations. The union of logic and algebra thus led to the development of digital computers and the rise of computer science in the mid-twentieth century. The connections between mathematics, logic, and computer science are now very tight and many-faceted.

Our interest in logic in this text is primarily two-fold. We will rarely be interested in the use of logic as a theoretical foundation for mathematics, though we will make some occasional comments on this program. Instead, we will mainly be interested in learning logic for what it can teach us about the craft of proof-construction. This is something that is valuable for both mathematics and computer science: both prove key results in their fields, and proof also plays a role in verifying the correctness of computer programs. As we pursue this proof-theoretic aim, however, we will also be learning valuable foundational material for computer science. This connection will become prominent in Chapter 7, where we consider the close connection between algebra, logic, and circuits. We will lay the basis for this here, however: Chapter 1 will explore the first component of mathematical logic, Propositional or Sentential Logic, which is the part of logic devoted to Boolean connectives.

To see how logic contributes to our understanding of mathematical proof, the rest of this section will focus on several related matters. We will first describe what proofs are and tell what they are good for. That will lead us to explain how logical rules of inference are involved and to see how this is connected with the notion of validity. To get a full picture of what valid inferences are, we will explore how truth values and the logical forms of statements enter into this whole process. Since we aren’t yet working within the context of a definite system of logic, we will discuss these things in a general way so we don’t get bogged down in specific details or tangled up in unfamiliar symbolism.

**The Concept of Proof in Mathematics**

A mathematical proof or demonstration consists of a sequence of logically connected propositions that show a certain result to be true because it follows from results already accepted as true. The first purpose of a proof, then, is to convince its readers or hearers that a conclusion is correct by showing that it is implied by the premises, which have already been granted.

The entire sequence of statements that make up a proof is termed an argument. An informal argument, such as is found in mathematics textbooks, usually contains key words like ‘since’, ‘thus’, or ‘therefore’ to indicate what is being taken as given and what is being concluded. The conclusion is said to be deduced from or derived from its premises.

Significant mathematical conclusions are usually called theorems. Lesser results are often called propositions, while those that follow almost immediately from another result are termed corollaries. As far as logic is concerned, however, no special terms are either needed or used for these different types of results; they are all conclusions of arguments grounded in premises.

Proofs are used by mathematical researchers in two ways: to verify conjectures or establish consequences for themselves, and to transmit these results to their colleagues. Expository works like textbooks communicate well known results by means of proofs, too. Diagrams, examples,
and additional remarks explaining the intuition behind the results or restating them in a less formal way are important for conveying the meaning behind the results and convincing an audience of their correctness, but the main vehicle of mathematical communication and the final arbiter of the truth of a proposition is its deduction from known results.

A proof is meant, however, not only to establish and communicate the correctness of a conclusion but also to explain why it is true. Proofs convince us because they demonstrate how results are related to things we already know. We’re usually interested in more than bare truth; we also want linkage and meaning. Proofs give us this. They also help us comprehend the significance of certain ideas and techniques because they’re the ones that keep popping up in proofs at key moments and in different situations.

Finally, proofs have organizational value. They enable us to systematize a field of thought like mathematics into an interconnected deductive whole. Proofs can thus also help the learning and recall of mathematical propositions because they show how complex results are related to simpler ones. One need not memorize mathematics as a collection of disconnected facts.

**Mathematical Proofs and Rules of Inference**

If the conclusion of an argument is incorrect, the argument can be faulty in at least one of two ways: either the premises of the argument are problematic, or else the reasoning involved is invalid. Whether the premises are true is a matter of mathematics, not logic. Our concern at the moment is primarily with the latter issue: what determines whether an argument is conclusive?

Deciding this in general is complicated by the fact that mathematical proofs come in many shades of completeness, depending upon context. Some proofs are very sketchy, highlighting only the main points of the derivation and leaving various intermediate conclusions and maybe even some premises for the reader to supply. A given degree of completeness may be quite appropriate for one audience but not another. Mathematicians already familiar with a given topic don’t want to be bored with finicky details they can easily fill in for themselves. Results that are well-known and that obviously apply to the situation at hand may be assumed without being mentioned. But for someone less familiar with the field, a more complete argument might be in order. Of course, there is a point of diminishing returns in fleshing out an argument. A completely rigorous argument that supplies every step and reason is usually counterproductive (except when that is the intent of the proof), for then the main outline of the deduction may disappear from view, suffocated by a mass of logical minutiae. Making an argument more rigorous may impede communication instead of promote it.

Nevertheless, in the final analysis, whether or not an argument is conclusive depends on its ability to be made rigorous. Proofs thought to be valid are sometimes discovered to be problematic when all the details are spelled out. Testing whether a proof is correct reduces in the end to seeing whether each step of a complete proof is legitimate. If an intermediate step has been legitimately inferred from earlier propositions, it should be possible to identify precisely which logical rule of inference justifies it. And if pressed, an explanation of why the rule of inference is sound should be given.

On the secondary school level, arguments are sometimes made rigorous by putting them into a two-column format. The first column contains the assertions of the argument, and the second column supplies the reasons for asserting them. Such a procedure is useful for pulling an argument apart into its component propositions, but as practiced, there is a measure of arbitrariness about it, since the reasons themselves are often instances of a proposition or axiom that could be asserted as a step in the proof (and so really ought to be in the left hand column, not the right). At times, though, it will be a logical rule allowing you to assert a line in the proof. For example, to write down a premise, you cite as your reason that it is ‘given’; to conclude one option from a pair of alternatives when you’ve ruled the other one out, you might cite something like ‘the only remaining alternative’.

1.1-4
A more consistent and thorough-going proof analysis would therefore proceed by placing all mathematical statements in the left column, using the right column simply for citing the logical rules of inference that permit such statements to be concluded from earlier ones. Following this procedure, you would then arrive at a completely rigorous derivation.

Under ordinary circumstances, as we noted above, such a detailed form of proof will be unnecessarily long and complex, and thus undesirable. But in order to become familiar with the main proof strategies in mathematics, it is important for you to know the wide variety of inference methods that exist. You should learn to recognize when and how they are used by others, even when they must be culled from below the surface of a proof. And you should know how to use them in various combinations to construct your own deductions. For this reason we will analyze and construct proofs in fairly full detail for a while, using a variation on the two-column format just described.

**Inference Rules and Valid Arguments**

Identifying sound rules of inference adequate for the task of deduction in any field of thought, mathematics included, is one of the main tasks of logic. A *deduction system* for logic thus consists of a collection of rules of inference chosen because they are rather immediate, fairly simple or widely used, and sound.

The *soundness* of the basic rules can be taken as self-evident, or it can be demonstrated. In contemporary logic, a rule of inference is considered *sound* if it always produces *valid* conclusions when applied to premises of the appropriate logical forms; that is, if the conclusion of an argument *logically follows* from its premises or is a *logical consequence* of the premises. We will assume this equivalence between validity and logical implication throughout: *a set of premises logically implies a conclusion iff* the associated argument is *valid*.

What, then, is a valid argument? Since we are trying to define this notion in order to explain what a sound rule of inference is, we obviously can’t turn around and say that a valid argument is one whose conclusion is inferred using sound rules of inference. That catches us up in a vicious circle. Moreover, we have no grounds automatically to suppose that all logical consequences can be proved from their premise sets. Being a logical consequence of a set of premises is surely something that is or is not the case regardless of whether it can be shown to be so by a valid argument. At the very least, the *definition* of logical implication ought to be independent of the process of deducing conclusions via a deduction system, even if all consequences are provable. Once the notion of validity/logical implication has been independently defined, we can go on to investigate whether a given deduction system is *complete*; that is, whether the available rules of inference suffice to prove all of the logical consequences of a set of premises.

So what is a valid argument? Here’s a definition going back to Aristotle: *a valid argument is one in which the conclusion necessarily follows from the premises.* It’s an argument where you’re stuck with the conclusion, logically speaking, once you accept the premises. How exactly does this happen? Well, if the conclusion contains information not contained in the premises, taken jointly, you can still wiggle out of admitting the conclusion; but if the conclusion is already implicit in the premises, you have no way to back out: the argument is valid. Thus, *an argument is valid iff the information contained in the conclusion is already contained in the premises.* This gives us an alternative characterization of validity and a somewhat more definite way of thinking about the necessity mentioned in the definition above.

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* We will use the standard abbreviation *iff* to mean *if and only if.*
EXAMPLE 1.1 - 1
Determine the validity of the following simple mathematical argument:
1) The square of any real number is non-negative;
2) The square of $\sqrt{-1}$ is $-1$, a negative number;
   therefore,
3) $\sqrt{-1}$ is not a real number.

Solution
This is a valid inference: the conclusion is a necessary logical consequence of the premises. Premise 1 tells us there is no overlap between the set of real numbers and the set of numbers whose square is negative, and the second premise tells us that $\sqrt{-1}$'s square is negative. Based on this information, we're forced to conclude that $\sqrt{-1}$ isn't a real number, whatever else it might be. If it were, its square would be non-negative, but it’s not.

Truth & Consequences, Round 1: A Necessary Condition for Validity

In the example just given, the premises and the conclusion were all true. However, this doesn’t need to be the case in order for an argument to be valid, as the next example shows.

EXAMPLE 1.1 - 2
Explain why the following premises logically imply the stated conclusion:
1) Every real number is either positive or negative;
2) 0 is a non-negative real number;
   therefore,
3) 0 is a positive real number.

Solution
This inference is valid because the conclusion necessarily follows from the premises. For, if 0 is a non-negative real number and the only options for real numbers are that they be positive or negative, then the first option must hold: 0 is positive. That the first premise is actually false is irrelevant to the argument’s validity; the conclusion logically follows from the premises. Knowing that the conclusion is false and that the argument is valid, however, we can conclude that one or more of the premises are false; in this case, just the first one.

So valid arguments can be made from all true premises, but they can also be made if some or all of the premises are false. You might think that the latter case is a silly one, because no one would want to argue from false premises. But this happens all the time. Sometimes people argue from results they mistakenly think are true but aren’t. Other times they argue from results they’re pretty sure are false in order to show that they are. This is a standard strategy in debates; it also happens in mathematics, where it’s called proof by contradiction. So valid inferences are not limited to ones in which the premises and conclusion are all true.

That being said, not every combination of truth values for premises and conclusion can be associated with a valid argument.

EXAMPLE 1.1 - 3
Determine the validity of the following argument:
1) All primes greater than 2 are odd;
2) 1001 is an odd number greater than 2;
   therefore,
3) 1001 is prime.
Solution
This argument is invalid. Both premises are true, but the conclusion is false: 1001 is not prime (factor it). The conclusion can’t possibly follow from the premises, then, for falsehood is never a logical consequence of truth.

If, as we claimed above, the information of a valid conclusion is already contained in its set of premises, and if the premises are true, then the conclusion can’t possibly be false: truth must yield truth if the inference is valid. Note what this does not say: it doesn’t say that any time a conclusion and its premises are all true, then the argument is valid, nor does it say that if an argument is valid then its premises and conclusion are true. It says instead: if an argument is valid, then, if the premises are true, the conclusion must also be true. Period.

NECESSARY TRUTH-VALUE CONDITION FOR VALID ARGUMENTS
If a valid argument has true premises, then its conclusion is true. Equivalently:
If an argument has true premises and a false conclusion, then it is invalid.

According to this principle, truth conditions give us a necessary condition for an argument to be valid. This helps us show that some (though not all) arguments are invalid, but it does not yet give us a criterion for judging an argument to be valid. Evidently, logical implication isn’t a simple truth-functional relation.

What does all this mean for inference rules? A rule of inference that would legitimize an inference in which the premises are true but the conclusion is false cannot be considered a sound rule and should be rejected. True statements do not have false consequences, so our inference rules should never permit us to pass from truth to falsehood, either. If they did, deductions would be incapable of grounding the truth of conclusions in the truth of their premises, and we would have no real motivation to deduce consequences from premises.

Truth & Consequences, Round 2: Validity and Logical Form

The truth-value connection to valid arguments is part of the story, but it is not yet the whole story. More than the actual truth values of the propositions in a given argument must be involved in defining the notion of “logical consequence” or “valid argument”.

What we are missing yet is a characterization of the intimate logical relation holding between a set of premises and its consequences. If a consequence logically follows from a set of premises, then this must be due to the logical character of the propositions involved and not to the non-logical content or particular meaning of the terms involved. The subject matter about which the propositions speak is largely irrelevant to the logical connection holding between the propositions (we’ll elaborate on this below); what really counts is their logical structure or logical form. This is known in logic as the Principle of Material Irrelevance, but it might better be called the Principle of Logical Form.

PRINCIPLE OF LOGICAL FORM
The validity of an argument is determined by the logical form of its premises and conclusion.

It is difficult to give a precise definition of the logical form or structure of a proposition or to distinguish sharply between logical and non-logical terms, but we can make a good beginning here. Logical terms include sentential linking words like ‘and’, ‘or’, ‘not’, ‘if-then’, and ‘is/are’; and quantifier words, such as ‘all’ and ‘some’. Non-logical terms are, in the case of mathematical statements, terms that refer to mathematical objects, their properties, and relations. This being the case, it is easier to say when two mathematical sentences have the same logical form than it is to say what their logical form is. If all that’s changed in a sentence are mathematical terms, then the new sentence has the same logical form as the old one.*

* Such a translation needs to respect term-types to be meaningful: object names substituted for object names, relation names for relation names, and so on.
EXAMPLE 1.1 - 4

Identify the logical and non-logical terms in the following valid argument:

1) All real numbers are positive, negative, or zero;
2) \( \sqrt{-1} \) is neither positive nor negative nor zero;

therefore,
3) \( \sqrt{-1} \) is not a real number.

Solution

Here the non-logical terms are:

1) real numbers, positive, negative, zero;
2) \( \sqrt{-1} \), positive, negative, zero;
3) \( \sqrt{-1} \), real number.

The logical terms are:

1) all, are, or;
2) is, neither, nor;
3) is, not.

The word ‘therefore’ indicates passage to the conclusion from the premises. While it is a logical term, it is not part of the logical structure of the individual sentences involved, so we left it off our list. (It is a logical term on the argument level, not the sentence level.) At this point we have no good means for showing the validity of this argument based on the logical forms involved, but it should be intuitively clear to you that it is, in fact, valid. Note, though, that the Necessary Truth-Value Condition for Valid Arguments is not violated: the premises are both true, and so is the conclusion. We could try to argue for its validity in terms of the information involved, but the logical structure of the sentences binding the information together is a bit more complex than before, and so it gets more difficult to see that the information in the conclusion is already contained in the premises.

Let’s work further with the Principle of Logical Form to see if we can develop a criterion for validity. According to this principle, the validity of an argument is independent of its non-logical content; the relation of logical implication does not depend on the particular non-logical meaning of the constituent propositions. Put in another, more useful way: two arguments whose sentences have exactly the same logical form must alike be valid or invalid. So, then, if we were to uniformly replace each non-logical term in an argument with other terms of the same sort, this new argument would have the same validity as the original argument.

This at least gives us a more flexible criteria for establishing an argument as invalid. In this context, the transformation process just described is called the method of counterargument or the method of counterexample. To show that a given argument is not valid, you transform it into one that violates the necessary condition for being valid; that is, into an argument where the premises are true and the conclusion is false. If you can do this, then the original argument is invalid, too. We thus have the following more general necessary condition for validity.

**GENERALIZED NECESSARY CONDITION FOR VALIDITY**

*If an argument is valid, then every argument in that same logical form with true premises must have a true conclusion.* Equivalently:

*If an argument has true premises and a false conclusion, then every argument in that same logical form is invalid.*

EXAMPLE 1.1 - 5

Show that the following argument is invalid by the method of counterargument.

1) \( 2 + 2 = 4 \)

therefore,
2) \( \sqrt{2} \) is irrational.
Solution

Transform the given argument by replacing the mathematical term ‘irrational’ with the term ‘rational’ and leaving everything else the same.

1) \(2 + 2 = 4\)
   therefore,
2) \(\sqrt{2}\) is rational.

This argument keeps the premise true, but its conclusion is now patently false. Therefore, the argument is invalid. The original argument, having the same logical form, must therefore also be invalid.

You may feel like you’ve just been hoodwinked with this solution. First of all, you may think it’s illegitimate to convert ‘irrational’ into its opposite. It’s not; we’ve merely replaced one mathematical content-word with another one of the same sort (they’re both properties of real numbers) to test for validity. If you prefer, though, you can replace ‘irrational’ with ‘negative’ and you’ll get the same conclusion: the argument is invalid. You will not be able to convert a valid argument into an invalid one merely by taking opposites: try it and see.

Secondly, you may have thought the argument really was valid. Even if you don’t want to claim this simply because both statements are true, realizing that one truth need not have any logical connection with another one, you may think they could be so connected in this case. Both statements belong to ordinary arithmetic and have to do with the number 2, and the first one is simpler than the second, so maybe the second follows in some complicated way from the first. But even if the first sentence were somehow involved in proving the second, this would only be the case in the context of many other premises also taken from arithmetic. The conclusion does not follow from the single premise given; our counterargument shows this conclusively. The method of counterargument/counterexample is a powerful logical tool for disproving things in mathematics.

The above examples show that the truth values of an argument’s propositions are related in some way to the argument’s validity, but somewhat loosely. An argument that has true premises but a false conclusion is patently invalid, as we’ve said, but other combinations of truth values for premises and conclusions seem to be compatible with an argument being either valid or invalid. In Example 1 we had a valid argument in which both the premises and conclusion were true. On the other hand, Example 5 had a true premise and a true conclusion, but the argument was invalid. Example 2 gave a valid argument having both a true and a false premise and a false conclusion. If the conclusion of that argument had been instead that \(-1\) is a positive number, we would have had the same truth values for all the sentences, but now the argument would have been invalid. Other combinations of truth values for premises and conclusions are also compatible with both valid and invalid arguments; these possibilities will be left for you to explore in more depth as an exercise (see Exercise 19).

Truth & Consequences, Round 3: Tarski’s Validity Thesis

The validity of an inference, then, is largely undetermined by the actual truth values of the particular propositions involved. Nevertheless, truth values turn out to be absolutely crucial for determining whether an argument is valid or invalid. How can this be?

In adopting the Generalized Necessary Condition for Validity, we have shifted our concern with validity to a higher level, from particular arguments to all arguments of the same form. This gets us closer to being able to decide whether arguments are valid, as we’ll now show.

Suppose an argument is invalid. Will there always be another argument of the same logical form in which the premises are true and the conclusion false? Is reality rich enough with potential counterexamples so that we can transform the non-logical content to make the premises true and the conclusion false? If this is the case, we will have a fool-proof method for determining invalid arguments: concoct a counterargument. Naturally, in order to obtain such a
counterargument, you may need to use quite a bit of creativity and insight, but that’s an issue related to human ingenuity; failure may not indicate a deficiency in the world.

Surprisingly enough, modern logic takes exactly this approach to invalidity and hence to validity. The generalized necessary condition for validity is also taken to be sufficient: an inference is valid if no counterarguments exist. This valid-by-default approach (valid unless proven otherwise) was stressed by the twentieth century Polish logician Alfred Tarski, so we will call this Tarski’s Validity Thesis.

**TARSKI’S VALIDITY THESIS**

*If the conclusion of an argument remains true whenever the premises are, under all possible transformations of the non-logical terms, then the argument is valid.*

Thus, an argument is valid if there is no translation of the content terms that turns all the premises true and the conclusion false. This criterion sounds like it would be impossible to apply, since it refers to all possible transformations, but we’ll find in various contexts that there are ways to satisfy it without considering arguments individually.

Tarski’s Validity Thesis combined with our earlier Generalized Necessary Condition for Validity provides us with a full-bodied implementation of the Principle of Logical Form. Premises logically imply their consequences based upon the logical forms involved and not the content. By considering all possible transformations of the given sentences, we make the actual content irrelevant. The thing that remains constant in all of these transformations is the logical form, which captures how the content terms are logically related regardless of their particular meaning. Truth values are thus important for establishing invalidity, but not simply on the level of the individual argument. They become determinative on the level of argument forms. A necessary and sufficient condition for arguments of the same logical form to be valid is that *none of them* ever lead from true premises to a false conclusion, regardless of the particular meaning of its non-logical terms.

We thus arrive at the following key definitions.

**DEFINITIONS**

1. **Logical Implication**: A set of premises *logically implies* a conclusion iff every transformation of the non-logical terms making the premises true also makes the conclusion true.

2. **Logical Consequence**: A conclusion is a *logical consequence* of a set of premises iff it is logically implied by them.

3. **Valid Argument**: An argument is *valid* iff the premises logically imply the conclusion; that is, iff every interpretation of the non-logical terms that makes the premises true also makes the conclusion true.

**Truth & Consequences, Round 4: Validity and Logical Form II**

We noted in our solution to Example 4 that it was a bit difficult to identify the argument there as valid because of the complexity of its logical structure. The same problem exists for any intricate argument. We don’t seem to have made much positive progress on this yet, even with our definition of validity; in fact, it seems to commit us to an impossible task. To show that a given argument is valid, we have to consider all arguments in that same form and show that their truth values behave as they should. How can this be done?

We’ll take our cue from algebra. When we want to show how to solve quadratic equations in general, to give an algorithm that works for all such equations, we use letters to stand for the coefficients so we can work with a typical case. We then solve it in terms of those letters.
We’ll do the same sort of thing here. Rather than examine arguments individually, we will consider them in a general way, using letters to stand for the content-words involved. The non-logical material thus gets abstracted without being erased, while the logical form of the sentences stands out more clearly. Using this general symbolic format, we will be able to argue that the conclusion follows from the premises. This can be done either in a positive way by considering how the information of the conclusion must relate to that of the premises, or in a negative way by showing that no counterargument can be constructed that would establish invalidity.

EXAMPLE 1.1 - 6
Formulate an abstract symbolic version of the argument in Example 4, and then establish its validity.

Solution
We will use letters that remind us of the terms used in Example 4, though this is certainly not necessary.
1) All \( R \) are \( P \) or \( N \) or \( Z \);
2) \( I \) is neither \( P \) nor \( N \) nor \( Z \);
therefore,
3) \( I \) is not \( R \).

Here we may see more easily why the argument is valid. We’re no longer distracted or influenced (possibly wrongly) by the actual content or by whether the sentences are true or false. We’re forced instead to focus on the logical relations between \( R, P, N, Z, \) and \( I \). The first premise says that all \( R \)’s are \( P \)’s, \( N \)’s, or \( Z \)’s. The second premise says that \( I \) isn’t any of these things. Well, then, \( I \) can’t be an \( R \) either. The information about the connection between \( R \) and \( P, N, \) and \( Z \), augmented by the information about \( I \)’s relation to these same three things, leads to the information contained in the conclusion. Alternatively, if the premises were true, there is no way the conclusion can be false, so no counterargument can be produced. Thus, the conclusion logically follows from the premises, and the argument is valid.

You may think that this way of establishing validity is still a bit nebulous, and you’d be right. We will have more precise and effective ways of making validity claims stick when we begin studying a particular system of logic. But the basic idea will be the same. Using some symbolic way to treat arguments in general (exhibiting the logical form of the argument), we will prove validity by showing that whenever the premises are true, so is the conclusion.

At the very heart of all the logical notions we have mentioned, therefore, there seem to lie the notions of true statement and of logical form. Thus, prior to studying valid arguments and proofs, we must investigate the varieties of logical forms that sentences can possess and learn when they can be true. Armed with this knowledge, we will be able to go on to determine what logical forms valid arguments have. We can then choose a system of sound inference rules that yield simple, valid arguments. On the basis of such a deduction system, we can at last analyze why a proof is conclusive and show how to construct chains of deductive reasoning. This may seem to be a rather lengthy process, but it’s the route we need to take. We start on this journey for the logical system known as Sentential Logic beginning in the next section.
EXERCISE SET 1.1

Problems 1 - 6: Premises and Conclusions
Identify the premises and conclusions in the following arguments. If some of these are assumed but not stated, make them explicit. If an argument contains an intermediate conclusion, functioning both as a premise and a conclusion, identify it as such.

1. The square of a real number is non-negative because it is either zero or it is positive.

2. Since composite numbers are those positive integers which can be factored into a product of numbers neither of which is 1, and since 1 is not a prime, the product of two prime numbers is composite.

3. All differentiable functions are continuous; \( f(x) = \sin x \) is differentiable; therefore \( f(x) = \sin x \) is continuous.

4. Corresponding parts of congruent triangles are congruent; if two sides and an included angle of one triangle are congruent respectively to two sides and an included angle of another, the triangles are congruent; \( \triangle ABC \) and \( \triangle DEF \) have sides \( AB \) and \( BC \) congruent to sides \( DE \) and \( EF \), and \( \angle B \) congruent to \( \angle E \); thus, \( \triangle ABC \cong \triangle DEF \), and so sides \( AC \) and \( DF \) are congruent.

5. Since \( \sin^2 x + \cos^2 x = 1 \) for any degree angle \( x \), \( \sin^2 0^\circ + \sin^2 1^\circ + \ldots + \sin^2 89^\circ + \sin^2 90^\circ = 45 \frac{1}{2} \), since \( \sin 45^\circ = \frac{\sqrt{2}}{2} \) and \( \sin x = \cos(90^\circ - x) \).

6. The number 846 is divisible by 6 because every number divisible by both 2 and 3 is divisible by 6. Moreover, 846 is divisible by 2 because 6 is divisible by 2; and it is divisible by 3 because \( 8 + 4 + 6 \) is divisible by 3, and if the sum of a number’s digits are divisible by 3, then the number is divisible by 3.

Problems 7 - 10: Brief Explanations
Explain the following in your own words.

7. What is circular reasoning or circular definition? Is there anything wrong with using them? Why do you think such circles are sometimes called “vicious circles”? Is there anything similar to this phenomenon in computer science?

8. What mathematics or computer science courses have you taken in which proof has played a significant role? How did they enter into the course? Did you have to construct any proofs of your own? How do you respond to the use of proof in mathematics and computer science courses?

9. What are the various functions of a mathematical proof? What ones seem most important? Why?

10. What is the difference between a proposition being a logical consequence of a set of premises and being a provable conclusion from such a set. Which notion is more basic, logical consequence or provability?

Problems 11 - 16: True or False?
Are the following statements true or false? Explain your answer.

11. If an argument is valid and its premises are true, then its conclusion is also true.

12. If an argument is valid and some of its premises are false, then its conclusion is also false.

13. If a valid argument has a false conclusion, then some of its premises must be false.

14. If an argument has true premises and a true conclusion, then it is a valid argument.

15. If an argument has false premises and a true conclusion, then the argument is invalid.

16. If an argument has false premises and a false conclusion, then the argument is invalid.

Problems 17 - 21: Argument Analysis
Analyze the following arguments.

17. Analyze the bickering logic of the following argument between two preschoolers. The setting is the middle of the kitchen floor one cold winter day.
   J (trying to pick a fight): “Fat and lumpy people don’t like to wear snow pants.”
   S (putting her snow pants on without complaint): “I’m not fat and lumpy.”
   J: “Yes, you are.”
   S: “No, I’m not, because I’m wearing my snow pants.”
18. State a valid argument of your own, from mathematics or everyday life, and explain why you think it is valid.

*19. Truth Values and Validity
   a. Draw up a table to indicate all possible truth-value combinations for an argument having two premises and a conclusion. Use separate columns for each premise and conclusion and fill in the entries with either T or F. How many different combinations of truth-value assignments are possible?
   b. For each truth-value assignment you made in part a (the different rows of your table) try to give two arguments of that type: one that is valid and one that is invalid. You may use arguments either from everyday life or from mathematics. When you’re done, tell which truth-value assignments, if any, are associated only with invalid arguments, and which ones, if any, are associated only with valid arguments. Explain.

20. Answer the following riddle: “Brothers and sisters have I none, but that man’s father is my father’s son. What is my relation to that man?” Then construct an argument for your answer, identifying the premises (whether assumed or stated) and your conclusion.

*21. Who’s Telling the Truth? Who’s Not?
   On a certain island, politicians never tell the truth, but everyone else does.
   a. A stranger meets three natives, and asks the first one if she is a politician. She mumbles an answer too soft to be heard. The second native says that the first one denies being a politician. The third native smiles and says that nevertheless, the first native is indeed a politician. Are any of these three natives politicians? How many politicians are there in this trio? Explain the reasoning behind your answer.
   b. A stranger meets three natives and asks how many of them are politicians. The first one says they all are politicians. The second one disagrees, claiming that only two of them are politicians. The third one walks away without answering. How many politicians are there? Explain your reasoning and speculate about the third native’s rude behavior.

EC c. A stranger walks along the main path until he comes to a fork in the road, where a native is sitting. What one question can the stranger ask to determine which path leads to the home of the island’s chief politician? What further question could he ask to discover whether the native is a politician or not?

Problems 22-23: Proofs and Rules of Inference

Read through the following sample proofs and see how many rules of inference you can identify. Since the proofs are written in typical abbreviated form, you may first want to write them out in more detail and put them into a two column format (assertion; reason). Then put all assertions in the left hand column (as suggested in the text) and reserve the right hand column for rules of inference. You will probably find this very difficult, since we haven’t identified many rules of inference to this point, but do what you can.

22. Theorem: If the hypotenuse and a leg of one right triangle are congruent respectively to the hypotenuse and a leg of another right triangle, then the two triangles are congruent.

   Proof:
   Let \( \triangle ABC \) and \( \triangle A'B'C' \) denote the two right triangles with right angles at \( A \) and \( A' \) and with leg \( AC \cong \text{leg } A'C' \) and hypotenuse \( BC \cong \text{hypotenuse } B'C' \).
   Extend line segment \( AB \) from \( A \) to a point \( D \) opposite \( B \) so that \( AD \cong A'B' \).
   Connect \( C \) and \( D \) to form \( \triangle ACD \).
Then by SAS, \( \triangle ADC \cong \triangle A'B'C' \).
Thus \( DC \cong B'C' \) and so \( DC \cong BC \), too.
\( \triangle BCD \) is therefore an isosceles triangle, so \( \angle B \cong \angle D \).
By AAS, \( \triangle ABC \cong \triangle ADC' \), and so \( \triangle ABC \cong \triangle A'B'C' \).

23. **Theorem:** The product of two odd integers is odd.

**Proof:**

Let \( m \) and \( n \) denote two odd numbers; say \( m = 2j + 1 \) and \( n = 2k + 1 \), where \( j \) and \( k \) are integers.

Then \( mn = (2j + 1)(2k + 1) \)
\[ = 4jk + 2j + 2k + 1. \]
\[ = 2(2jk + j + k) + 1. \]

But this is an odd number, since it is 1 more than an even number. ■
2. The word 'since' usually indicates the following clause is a premise.

4. There is an intermediate conclusion in this argument.

6. ‘Because’ usually indicates the following clause is a premise.

8. [No hint.]

9. See pages 3 and 4 for some ideas on this one.

11. [No hint.]

12. [No hint.]

13. [No hint.]

14. [No hint.]

19. a. [No hint.]
   b. You should be able to come up with both valid and invalid arguments for every truth-value assignment but one.

21. a. Everyone, politicians and non-politicians alike, will give the same answer when asked whether or not they are a politician (What answer do they give?). Knowing this, you should be able to tell whether or not the second native is telling the truth or not.
   b. From the first native’s statement, you can tell two things. (1) There are no more than two politicians. (2) The first native is a politician. Explain how these statements can be concluded.
Mathematical logic has two main parts. The first is known as Propositional or Sentential Logic (SL). It deals with matters that depend on how whole sentences form more complex sentences using connectives like “or” and “if-then”. The other part of logic deals with deeper internal structure; it focuses on terms and predicates along with the logical quantifiers “all” and “some”. The full system of logic is known as Predicate Logic (PL).

This chapter looks at Sentential Logic; Predicate Logic is the focus of Chapter 2. Though SL is simpler than PL, it is still a very important system of logic. Studying SL pays large dividends to anyone who wants to learn basic proof strategies for mathematics. SL is also the part of logic most closely related to algebra and computer science.

We will begin by looking at some simple ways in which sentences combine to make compound sentences. We will then consider valid arguments insofar as they depend on inter-sentential logical structure. Once we understand how to determine valid arguments, we will be able to formulate a number of inference rules for SL’s deduction system and use them to construct deductions.

We will present Sentential Logic as an abstract or formal system of logic. We will regularly give mathematical or everyday examples to illustrate our discussion, but we will usually formulate our sentences and arguments symbolically, using letters. As noted in the last section, this will make the logical forms of the sentences and arguments clearer.

Sentences and Truth Values

In ordinary speech a sentence is a complete statement formed according to the rules of grammar (syntax) and communicating a meaningful thought (semantics). We use various types of sentences in everyday discourse: questions, commands, declarations, and so on. The only sorts of sentences we will formalize with SL are declarative statements, ones that affirm or deny something to be the case.

Such sentences are true in case what they say is actually so and are false otherwise. This correspondence notion of truth gets at the concept of accuracy. While there may be different and deeper meanings of the term in other contexts, it is this notion of propositional truth, going back to Plato and Aristotle, that is used in mathematics. It also underlies the modern approach to symbolic logic elaborated by Tarski and others. A sentence is true iff what it asserts is the case, and it is false iff what it asserts is not the case.

Since a given state of affairs either is or is not the case and cannot be both, a sentence is either true or false but not both. A sentence has a unique truth value, which is either ‘True’ (abbreviated by T or 1) or ‘False’ (abbreviated by F or 0). SL is thus a two-valued system of logic.

Syntax and Semantics of SL

Natural languages provide many kinds of connectives or operators for combining sentences. The only ones allowed in SL, however, are truth-functional connectives. A sentential connective is a truth-functional connective iff the truth value of the compound sentence it generates is uniquely determined by the truth values of the component sentences.

The main truth-functional connectives studied in SL are “and”, “or”, “not”, “if-then”, and “if and only if”. A sentence that contains one or more of these operators is a compound sentence; otherwise it is considered an atomic or primitive sentence, no matter how long or complex it may look. On the other hand, a sentence that seems rather simple may still be compound; its truth-functional connectives may be hidden by an abbreviated formulation typical of ordinary
language. Recognizing precisely which sentences are compound and which ones are atomic comes with practice in analyzing the underlying logical structure of sentences.

\textbf{EXAMPLE 1.2 - 1}

Determine which of the following mathematical propositions are atomic sentences. For those that are compound, identify their atomic components and the logical connectives that are involved.

a. $1 + 2 + 3 + \cdots + 99 + 100 = \frac{100 \cdot 101}{2}$.

b. $|\pi - \frac{22}{7}| \geq 0$.

c. If $p$ is a prime number greater than 2, then $p$ is odd.

\textbf{Solution}

a. This first sentence is atomic; it contains no subsentences, just lots of terms, most of them missing but indicated by the dots.

b. The second sentence is much shorter, but it is compound. The atomic components are:

\[ |\pi - \frac{22}{7}| > 0 \]
\[ |\pi - \frac{22}{7}| = 0 \]

The only logical connective appearing here is ‘or’; it is understood as embedded in the mathematical notation ‘\( \geq \)’.

c. Sentence three is more compound than it looks. The only obvious connective is “if-then”, but the first clause is actually a conjunction. The atomic components are:

- $p$ is a prime number
- $p$ is greater than 2
- $p$ is odd

The connectives involved here are “if-then” and “and”. Formulated in a way that reveals the full compound structure more clearly, the sentence reads: If $p$ is a prime number and $p$ is greater than 2, then $p$ is odd.

To symbolize sentences in SL, we will use capital letters/constants such as $P$, $Q$, $R$, and $S$ to stand for specific whole sentences (not for parts of sentences, as in Section 1.1). Letters standing for sentences in general (sentence variables) will be put in bold face font: $\mathbf{P}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{S}$. Letters may represent either atomic or compound sentences; a single letter only indicates that a complete sentence is being denoted, not that it is atomic. In addition to letters, we will use special symbols to stand for the various logical connectives. To represent a compound sentence of SL, we will write a string of symbols containing both letters and connective symbols. Right and left parentheses (and occasionally brackets and braces) will be used in place of ordinary punctuation, both to guard against ambiguity in our symbolic formulas and to make them more readable.

We will explain the precise grammar or production rules for writing such sentences as we proceed: this will constitute the \textit{sentence syntax of SL}. When a string of symbols is formulated in accord with the syntactic rules of SL, it is called a \textit{sentence} or \textit{well-formed formula} of SL. We will also indicate how such symbolic sentences are to be interpreted and how truth values are to be assigned to them: this comprises the \textit{sentence semantics} of SL.

In this lesson we will investigate the syntax and semantics for \textit{conjunction}, \textit{disjunction}, and \textit{negation}. We will also define and illustrate the notion of logical truth and its opposite, logical falsehood. Semantic notions associated with arguments will be taken up in Section 1.3.
## Syntax and Semantics of Conjunction

The connective “and” is the simplest and least controversial of all sentential connectives. It joins two sentences “P” and “Q” to form the conjunction “P and Q”. For brevity, we will symbolize “and” by ∧, which can be thought of as the outline of a capital A, the first letter of ‘And’. We will use ordinary in-fix notation for our binary operators in SL; thus the sentence ‘P and Q’ is symbolized by P ∧ Q. Some logic texts use the ampersand ‘&’ instead, which might seem a more natural choice (and more fun to write), but this notation lacks an appropriate counterpart for “or”. Other books use the dot of ordinary multiplication; we will introduce this when we look at the connection between SL and Boolean Algebra in Chapter 7.

### DEFINITION 1.2-1: Truth-Value Assignment for Conjunction

A conjunction P ∧ Q is true iff P is true and Q is true.

This certainly seems to be the appropriate truth-functional definition for “and”. For if a conjunction “P and Q” were the case, surely P would be the case, as would Q. We can summarize this truth-value assignment for conjunction by giving a truth table (devised by the late nineteenth-century American mathematical logician and philosopher, C. S. Peirce). A truth table exhibits all possible combinations of truth values for the sentence variables P and Q and gives the truth value of the conjunction P ∧ Q for each assignment. We can do this in full form, using headings for P, Q, and P ∧ Q, and placing the possible truth values below them. Or, following the twentieth-century American logician Quine, we can write down a compact form, putting the truth values of the conjuncts directly below the letters and the truth value of the full conjunction below the ∧. The final column of truth values representing the whole sentence will often be highlighted by underlining it one or more times to make the table more readable.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>P ∧ Q</th>
<th>P ∧ Q</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T T T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>T F F</td>
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<tr>
<td>F</td>
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<td>F</td>
<td>F F T</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F F F</td>
</tr>
</tbody>
</table>

For future reference, note how truth values have been assigned in rows to the constituents P and Q: the final right-hand letter column alternates T F T F, while the first letter column has all T first and then all F. This seems to be the most popular way to take care of all four possible truth-value assignments, though texts that use 0 for F and 1 for T often reverse the order to coincide with numerical order. To make comparisons between your work and others’ easier, you should follow this procedure in constructing all your truth tables for formulas with two sentence variables.

The word ‘and’ is not always used as a sentential connective in ordinary speech. At times it indicates a list of events or items joined together, sometimes in sequential order. The sentence ‘Brian and Rick moved the refrigerator back into place’ uses ‘and’ in such a way. We cannot expand this sentence into ‘Brian moved the refrigerator back and Rick moved the refrigerator back’; both were needed to move it. A similar thing can be said about the sentence ‘The product of 2 and 3 is 6’. The sentence ‘There’s the windup and the pitch’ uses ‘and’ to combine two sentences, but it’s more than a simple truth-functional connective.* It means ‘there’s the windup and there’s the pitch’, but it also conveys an order to the actions that a truth-functional connective doesn’t.

---

* SL doesn’t perfectly match ordinary language usage, but it’s adequate for deductive reasoning in mathematics and elsewhere.
On the other hand, as we’ve already seen in Example 1.2-1, conjunction is sometimes present where ‘and’ is missing. Ordinary language often piles up adjectives as an abbreviated form of conjunction. The mathematical sentence ‘\( \triangle ABC \) is a right isosceles triangle’ can be expanded into the conjunction ‘\( \triangle ABC \) is a right triangle and \( \triangle ABC \) is an isosceles triangle’.

‘The number 2 is prime but not odd’ asserts in shortened form that ‘the number 2 is prime and the number 2 is not odd’. The word ‘but’ is often used as a conjunctive where some sort of opposition is being stressed. In translating such sentences, however, we will use ‘and’ as our connective.

There also are some mathematical formulas in which “and” is camouflaged in the mathematical notation. This is the case with \( a \leq b \leq c \), which means \( (a \leq b) \land (b \leq c) \). In such cases we will usually not stoop to reformulating the formula using \( \land \); we will leave the implicit “and” buried in the notation. It is important, nevertheless, to understand that such formulas are really conjunctions if they are to be used properly in arguments.

### Syntax and Semantics of Disjunction

The connective “or” is our second binary connective. The result of joining two sentences \( P \) and \( Q \) by the word ‘or’ to obtain \( P \lor Q \) is called the disjunction of \( P \) and \( Q \). We will symbolize “or” by \( \lor \); ‘\( P \lor Q \)’ is written in abbreviated form as \( P \lor Q \).

As with “and”, there are exceptions to using this disjunction symbolism in certain mathematical contexts where an already accepted notation exists, such as \( a \leq b \), which is short for \( (a < b) \lor (a = b) \), or \( x = \pm 1 \), which abbreviates \( (x = +1) \lor (x = -1) \).

Exactly what the most appropriate interpretation of logical disjunction should be is less fixed by ordinary discourse than that of conjunction. At times, “or” seems to be meant in a narrow, exclusive sense. At first thought, this seems to be the case whenever two mutually exclusive alternatives are present. The sentence

\[
\sqrt{2} \text{ is rational or } \sqrt{2} \text{ is irrational}
\]

seems to use ‘or’ in this narrow sense. At any rate, both of the disjuncts cannot be true. In this case and others like it, however, the two sentences are not logically unrelated; truth values cannot be assigned to the disjuncts independently; one is the negation of the other. The subsentence ‘\( \sqrt{2} \) is irrational’ means ‘\( \sqrt{2} \) is not rational’; its truth value obviously depends on that of ‘\( \sqrt{2} \) is rational’. Thus the meaning of the disjuncts precludes the possibility of taking them to be true simultaneously. Whether or not “or” is exclusive, therefore, cannot be tested by means of such specialized examples. It must be possible to assign two ‘True’ values to the disjuncts in order to decide whether “or” is being used in an exclusive sense or not. Given any other truth-value assignment, the truth value of the disjunction turns out to be identical both for the exclusive sense and the non-exclusive sense of “or”, to be discussed shortly.

There are cases where exclusive “or” is definitely intended. Duane says, “I’ve done just about enough studying for one night. Either I’ll finish my logic assignment, or else I’ll study some statistics, but then I’m off to bed.” Here there is no logical reason why both disjuncts cannot be the case, but the ordinary intent of such a statement is such that if Duane studies both subjects before going to bed, then he spoke falsely about his intentions. The ‘or’ used on a restaurant menu in saying you may have either a salad or a vegetable with your meal is also an exclusive “or”.

On the other hand, there are times when ‘or’ is used in a broader, non-exclusive sense. “I have some money in my wallet or back in my room” ought to be considered a true statement even if I have money in both places. Such a truthful use of ‘or’ does not exclude both alternatives from being the case. Non-exclusive ‘or’ is also the type of disjunction wanted and used in Boolean searches on the internet and elsewhere.
Which connective to make primary is thus largely a matter of choice. Fortunately, there is no need to choose one at the expense of the other. Logic has room for both, even if one is taken as the preferred reading for the symbol \( \lor \). As a matter of fact, Sentential Logic has been developed in both ways. Following an earlier tradition, George Boole used “exclusive or” in his system of logic. The modern approach, though, follows Boole’s successors Jevons and Peirce in adopting the non-exclusive meaning for “or”, since it lends itself more easily to algebraic treatment. It is also the simpler of the two, in the sense that exclusive disjunction can be defined more easily in terms of it than the other way around (see Exercise Set 1.3). Furthermore, treating “or” as a non-exclusive connective yields simpler logical equivalents involving disjunction for familiar formulas. This, too, argues in favor of adopting a non-exclusive interpretation. Finally, “non-exclusive or” makes it slightly easier to formulate and reason with certain mathematical propositions, such as statements of set theory that involve set union. For all these reasons, we will assume that “non-exclusive or” is intended in a mathematical sentence and formulate it with \( \lor \).* We will use another symbol, \( \lor \) or XOR, to represent “exclusive or” when it is wanted.

**DEFINITION 1.2-2: Truth-Value Assignment for Disjunction**

A disjunction \( P \lor Q \) is true iff either \( P \) is true or \( Q \) is true.

The truth table for \( P \lor Q \) is the following:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th>( P \lor Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
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<tr>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>

There is one other way in which ‘or’ is sometimes used in informal mathematical writing and ordinary speech. Here ‘or’ is used to indicate that what follows is an equivalent or alternative formulation of the preceding term or clause. This practice treats ‘or’ as a term similar to ‘i.e.’ or the logical connective ‘iff’ (see Section 1.4). Since there are perfectly good alternatives to using ‘or’ in this way, you should try to avoid it and reserve ‘or’ for expressing non-exclusive disjunction.

**Syntax and Semantics of Negation**

The sentential connective “not” differs from the two previous connectives. In fact, you might question whether it even is a connective, since it doesn’t do any connecting. It is a sentential operator, however; a *unary*, rather than a binary, operator. Given a sentence \( P \), the operator ‘not’ applied to this sentence turns it into its logical opposite, ‘not-\( P \)’. If \( P \) is the sentence ‘\( \sqrt{2} \) is rational’, then its negation ‘not-\( P \)’ is the sentence ‘\( \sqrt{2} \) is not rational’. The word ‘not’ is usually incorporated somewhere in the middle of an ordinary sentence, seemingly negating the verb or predicate clause, but it is actually the entire statement that is being negated.

The hooked minus sign \( \neg \) seems to be the most popular sign in mathematical circles for logical negation. Some textbooks use a tilde \( \sim \) (a squiggly ‘n’?'), but since this symbol has other uses in mathematics as a relation symbol while \( \neg \) doesn’t, we will use the hook. The negation of \( P \), then, is \( \neg P \), read as ‘not-\( P \)’. A more elaborate reading, but one that can be

* As an aside: the two words in Latin for “or” that correspond to the exclusive and non-exclusive senses are ‘aut’ and ‘vel’ respectively. Whether or not this is the reason why \( \lor \) (standing for ‘vel’?) was chosen to denote “non-exclusive or”, one would like to believe that some such reason lies behind the symbolism.
used to negate ordinary sentences in a uniform way, is ‘it is not the case that \( P \)’. This sounds a bit awkward, but it makes it clear that the entire sentence is being negated. On this reading, the above negation becomes ‘it is not the case that \( \sqrt{2} \) is rational’.

In certain mathematical contexts it is convenient to use the slash sign \( / \) to indicate negation. There are quite a few standard mathematical notations that already incorporate negation in this way. A statement that two objects are not related to one another by a certain relation is shown simply by crossing through the relation symbol in a sentence that would otherwise assert their relation. This is what is done, for instance, in cases like the following:

\[
0 \neq 1; \quad \triangle ABC \not\equiv \triangle A'B'C'; \quad x \notin S.
\]

We will continue to follow this practice rather than rewriting sentences in expanded form by pre-fixing a \( \neg \), which makes them less easy to read:

\[
(0 = 1); \quad (\triangle ABC \equiv \triangle A'B'C'); \quad (x \in S).
\]

In cases where there is no established symbolism for negating sentences, however, we will use the negation hook. And we will naturally use it to negate sentences formed using letters as we formulate general results about negations.

Since sentences are either true or false but not both, when you negate a sentence, you change its truth value into the other value. This gives us the following truth-value assignment for negation.

**DEFINITION 1.2-3: Truth-Value Assignment for Negation**

A negation \( \neg P \) is true iff \( P \) is false.

The truth table for \( \neg P \) is the following:

<table>
<thead>
<tr>
<th>( P )</th>
<th>( \neg P )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

Since “not” is a sentential operator, it can be applied to any sentence whatsoever, even sentences that are already negations. The meaning of a simple negation is quite clear, but multiple negations are often abused in everyday discourse. A double negative, for instance, is sometimes thought to indicate a strongly felt negation. In two-valued logic, however, negation is negation and our feelings about it don’t count. Detroit Tiger manager Sparky Anderson could approve of his pitcher Doyle Alexander with the words, “I love a guy where I don’t never have to worry about no ball fours” (August, 1987) and mean that he never had to worry about Alexander giving up a walk, but that’s not what the sentence strictly means. In logic, double negation is the logical opposite of negation and so amounts to asserting the original statement in an equivalent form: \( \neg\neg P \) and \( P \) have exactly the same truth tables.*

* There are systems of logic, however, such as intuitionist logic, where a double negation is considered weaker than the original proposition. This leads to a different system of deduction, since it affects what one can conclude when negations are involved. We will occasionally note some of these differences as we go through SL.
### Complex Compound Sentences

Using $\land$, $\lor$, and $\neg$, we can make highly complex propositions out of simple ones. The semantics of such compound sentences is fairly straightforward as long as the syntax is unambiguous; that is, as long as it is clear how the sentences have been constructed from the various atomic sentences by the sentential connectives. Without some sort of priority convention, however, there can be a good deal of confusion.

For example, consider the following compound mathematical sentence, which could come up in connection with solving a quadratic equation for a positive root:

\[
(*) \quad x \geq 0 \land x - 1 = 0 \lor x + 2 = 0.
\]

Just what does this mean? Is $x = -2$ a solution or not? Well, it all depends on the order in which the sentential connectives operate on the parts. If $\land$ is the last connective applied (probably the intended meaning), then $x = 1$: the solution $-2$ must be ruled out because it is negative. On the other hand, if the final connective is $\lor$, $x = -2$ is still a solution.

Suppose you are given an even simpler abstract sentence: $\neg P \lor Q$. What sort of sentence is this, a negation or a disjunction? It depends on how it was constructed. $P$ and $Q$ could have been combined into a disjunction $P \lor Q$ and then the result negated. Or $P$ could have been negated and then $\neg P$ and $Q$ joined in a disjunction. These give two completely different sentences, semantically as well as syntactically, and so they must be distinguished.

There are several ways in which sentences can be made single-valued. A parenthesis-free notation, devised by twentieth century Polish logicians, puts operator symbols on the left of the letters operated upon (pre-fix notation) and the requisite number of sentence letters on the right in the correct order. This way seems unnatural to someone accustomed to the algebraic practice of putting operation symbols between the letters (in-fix notation), but by means of it you can make sentences completely unambiguous. The above sentences would be written, using Polish Notation (but our connective symbols), in the following way: not $P$ or $Q$ as $\neg P \lor Q$, and not $P$ or $Q$ as $\neg (P \lor Q)$ respectively.

A more common way in which formulas can be made unambiguous is by using parentheses as needed to indicate which sub-results should be joined into a single unit. Parentheses indicate, as in algebra, that the operation inside them must be performed before doing anything further with that result. Here ‘not $P$-or-$Q$’ would be written as $\neg (P \lor Q)$, and ‘not $P$ or $Q$’ as $(\neg P) \lor Q$.

However, as in algebra, it is possible to stipulate some operations as being stronger than (taking precedence over, having priority over) others. We can thus save on some parentheses. Following the usual convention, we will take $\neg$ to have highest priority; it takes precedence over (gets applied before) both $\land$ and $\lor$. Of $\land$ and $\lor$, $\land$ takes priority over $\lor$ (like multiplication over addition in algebraic formulas).

<table>
<thead>
<tr>
<th>Priority</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\neg$</td>
</tr>
<tr>
<td>2</td>
<td>$\land$</td>
</tr>
<tr>
<td>3</td>
<td>$\lor$</td>
</tr>
</tbody>
</table>

With this convention, ‘not $P$ or $Q$’ can be written without any parentheses as $\neg P \lor Q$: parentheses must still be used for ‘not $P$-or-$Q$’, writing it as $\neg (P \lor Q)$. When parentheses are dropped from longer formulas according to the priorities just set up, the formulas become unambiguous, but the formula may still be misread due to its complexity. To guard against this and make such formulas more readable, we will occasionally include parentheses where they could be omitted.
EXAMPLE 1.2-2

Draw up truth tables for the two sentence forms \( \neg (P \vee Q) \) and \( \neg P \vee Q \).

Solution

The truth tables for these sentences are as follows:

\[
\begin{array}{ccc}
P & Q & P \vee Q \neg (P \vee Q) \\
T & T & T & F \\
T & F & T & F \\
F & T & T & F \\
F & F & F & T \\
\end{array}
\]

\[
\begin{array}{ccc}
P & Q & \neg P & \neg P \vee Q \\
T & T & F & T \\
T & F & F & F \\
F & T & T & T \\
F & F & T & T \\
\end{array}
\]

The syntactic structure of a SL sentence reflects the way in which the sentence is produced from its atomic sentences. This can be illustrated nicely with a production graph, starting with the atomic sentences on the bottom, and proceeding up the graph to more compound components, until the full sentence is at the top of the graph. For instance, the production graphs for the two sentences in Example 2 would be the following.

\[
\begin{array}{c}
\neg (P \vee Q) \\
P \vee Q \\
P & Q \\
\end{array}
\]

\[
\begin{array}{c}
\neg P \\
\neg P \vee Q \\
P & Q \\
\end{array}
\]

The connective that is last used to join the various parts of a compound sentence together is known as the principal or main connective of the sentence. The sentences that it connects are the main subsentences of the sentence. It is important for you to recognize what the main connective is for any given formula, both for working with abbreviated truth tables and, more importantly, for constructing deductions later on, for it determines what sort of sentence the compound sentence is overall. In the two sentences just given, the main connectives are \( \neg \) and \( \vee \) respectively.

EXAMPLE 1.2-3

Determine the main connective of the sentence form \( \neg P \wedge (Q \vee R) \), which is one way to represent the mathematical sentence (*) above, taking \( x \geq 0 \) to mean \( x \not< 0 \). Then write down its abbreviated truth table.

Solution

The main connective for the sentence \( \neg P \wedge (Q \vee R) \) is \( \wedge \). The truth-value assignment for the entire sentence can be read off below this connective (above the double underline). The single underlines indicate main connectives of inner-level component sentences whose values must be calculated first.
$\neg P \land (Q \lor R)$

<table>
<thead>
<tr>
<th></th>
<th>T</th>
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</table>

Note that since three sentence letters are involved here, the truth table must have eight lines in it to account for all the possible truth-value combinations they can be assigned.

The last letter in the formula ($R$) again has truth values that alternate $T$ $F$ $T$ $F$, while the letters to its left ($Q$ and $P$) have truth values that alternate in blocks of two or four.

**Logical Truths and Falsehoods**

There are three distinct types of compound SL sentence forms, so far as truth values are concerned. Most sentence forms can take on both $T$ and $F$ when various truth-value assignments are made for the sentence symbols (the atomic sentences) involved. However, there are also sentences whose truth value is independent of the particular assignment given. These sentences may be always true or always false.

**DEFINITION 1.2 - 4: Logical Status of Sentences**

a) A logically-indeterminate sentence form is one that is both true and false, under different truth-value assignments.

b) A logically-true sentence form is one that is true under all truth-value assignments. Such logical truths are also called tautologies.

c) A logically-false sentence form is one that is false under all truth-value assignments. Such logical falsehoods are also called contradictions.

The first type of sentence is termed logically indeterminate because its truth value cannot be determined until the precise truth value of the constituent sentences is specified. The sentence forms considered in Examples 2 and 3 are all logically indeterminate. Logical truths and falsehoods are so named because they are true or false solely on the basis of the logical structure or form of the sentence. Regardless of the meaning of the sentences involved (and thus of the truth-value assignment), the compound sentence is always true or always false. You will see examples of these two types of sentences in a moment.

Among the class of logical truths, some involve only the sentential connectives “and”, “or”, and “not”. The two most important ones are associated with principles that have long been considered Fundamental Laws of Logic. The Law of Non-Contradiction, abbreviated $LNC$, captures the idea mentioned in the introduction to this lesson that something cannot both be the case and not be the case. Thus, if $P$ stands for any proposition whatsoever, $\neg(P \land \neg P)$ is a logical truth, as a short truth table will verify (see Exercise 41). This law is also called the Law of Contradiction by some authors.

The Law of Excluded Middle ($LEM$) states that something either is or is not the case; a third or middle alternative is excluded. Either a sentence is true or its negation is. (This law is disputed by intuitionist mathematicians.) These tautologies are of the form $P \lor \neg P$. Again, such sentences can be shown to be logical truths by means of a simple truth table, and will be left as an exercise (see Exercise 42).
There are other types of sentences as well that are tautological; we will meet up with some of them when we discuss the other sentential connectives, and also when we discuss rules of inference.

**EXERCISE SET 1.2**

*Problems 1-4: Sentence Symbolization*

Determine all atomic sentences within the given statements. Symbolize each sentence with a letter, providing an interpretation key, and then rewrite the entire sentence using logical symbolism.

*1. Either this is an easy exercise, or there’s more to it than I know.*

*2. Little Bo Peep has lost her sheep and doesn’t know where to find them.*

3. This, too, will pass.

4. Either this is more complex, or I’m mistaken; but I’m not mistaken.

*Problems 5-8: Sentence Symbolization*

Determine all atomic sentences within the given mathematical statements. Symbolize each sentence with a letter, providing an interpretation key, and then rewrite the entire sentence using logical symbolism.

*5. \(\triangle ABC\) is equilateral or isosceles or scalene.*

*6. \(n\) is divisible by 6 or \(n\) is not divisible by 2.*

7. \(\sin^2 \alpha + \cos^2 \alpha \leq 1.\)

*8. \(r > 0\) and \(|r| = r\), or \(r \neq 0\) and \(|r| = -r\).*

9. Give a strict interpretation of Sparky Anderson’s multiply negative statement about Doyle Alexander (see page 1.2-6). Do you think he really meant to say this? Reformulate the statement in logically proper English so that it says what he probably meant to say.

*Problems 10-16: True or False*

Are the following statements true or false? Explain your answer.

10. Meaningful particular sentences formulated via SL have exactly one truth value.

*11. ‘\(P \lor Q\)’ is true only if one of ‘\(P\)’ or ‘\(Q\)’ is true.*

12. ‘\(\neg \neg P\)’ has the same truth table as ‘\(P\)’.

13. If the priority conventions for ‘\(\neg\)’, ‘\(\land\)’, and ‘\(\lor\)’ are used, parentheses are never needed for formulating SL sentences involving these connectives.

14. The main connective in ‘\(\neg P \land Q\)’ is ‘\(\land\)’.

*15. Logic alone can never determine the truth value of a sentence.*

EC 16. This statement is false. (This is a version of the so-called ‘liar paradox’.) Is this statement even a sentence, as we have defined the term for SL?

*Problems 17-19: Defining Terms*

Give the meaning for each of the following terms, putting it into your own words.

17. Truth functional connective

18. Main connective

*19. Tautology

20. Contradiction

1.2-10
Problems 21-24: Truth Values of Sentences
Design compound sentences involving the variables given to satisfy the stated condition.

21. A sentence involving \( P \) and \( Q \) is true iff neither \( P \) nor \( Q \) is true.
22. A sentence involving \( P \) and \( Q \) is true iff either \( P \) or \( Q \) are false.
*23. A sentence involving \( P \) and \( Q \) is true iff \( P \) and \( Q \) have different truth values.
24. A sentence involving \( P \), \( Q \), and \( R \) is true iff exactly one of \( P \), \( Q \), and \( R \) is true.

Problems 25-27: Problematic Formulations
What is wrong with the following statements, which are sometimes found in texts on logic and proof? What should be said instead?

*25. \( P \land Q \) means that \( P \) is true and \( Q \) is true.
26. \( P \lor Q \) says \( P \) is true or \( Q \) is true or both are true.
27. \( \neg P \) is an abbreviation for \( P \) is false.

Problems 28-31: Sentence Construction, Main Connectives, and Truth Tables
Indicate how each of the following compound sentences is produced from its atomic sentences by giving a production graph and identifying the main connective. Then write out its truth table.

28. \( P \lor \neg Q \)
29. \( \neg(P \land \neg Q) \)
*30. \( P \lor (\neg Q \land \neg R) \)
31. \( (\neg P \lor Q) \land R \)

Problems 32-35: Polish Notation
Write each of the following sentences in Polish pre-fix notation, using the given symbols for the connectives. Where does the main connective show up in the formula?

32. \( P \lor \neg Q \)
33. \( \neg(P \land \neg Q) \)
34. \( (\neg P \lor Q) \land R \)
EC 35. \( P \lor (\neg Q \land \neg R) \)

36. Is the sentence form \( P \land Q \land R \) ambiguous or not? Explain your answer carefully.

Problems 37-40: Logical Status of Sentences
Are the following sentences logically true, logically false, or logically indeterminate? Why?

*37. \( (P \lor \neg Q) \land (\neg P \lor Q) \)
38. \( P \lor \neg(P \land Q) \)
39. \( (P \land \neg Q) \land (\neg P \lor Q) \)
*40. \( (\neg P \land \neg Q) \lor (P \lor Q) \)

Problems 41-42: Tautologies
Show that the following laws are tautologies.

41. The Law of Non-Contradiction: \( \neg(P \land \neg P) \)
*42. The Law of Excluded Middle: \( P \lor \neg P \)

43. Explain why the negation of a tautology is a contradiction and why the negation of a contradiction is a tautology. Given this correspondence, what can you say about the total number of logical truths as compared with the number of logical falsehoods?
44. A truth table for a formula containing one sentence variable has only two lines. A truth table for a formula containing two sentence variables has four lines. A truth table for a formula containing three sentence variables has eight lines.
   a. Explain why this is so.
   b. How many lines would a truth table have for a formula containing four sentence variables? Support your answer.
   c. How many lines would a truth table have for a formula containing \( n \) sentence variables? (Generalize from the pattern you see emerging from parts a and b.) Support your answer with a proof as best you can.

*45. Find the solution set of all ordered pairs \((x, y)\) such that \( x(1 - y^2) = 0 \) and \((x + 2)y = 0\). Factoring each equation, work your solution step by step and point out precisely where the logical connectives ‘or’ and ‘and’ enter the process. Check your answers and explain.
HINTS TO STARRED EXERCISES 1.2

1. This is a disjunction of two atomic sentences.
2. There are two connectives in this sentence.
3. This sentence is short, but it contains three atomic sentences.
4. Don’t forget about negation.
5. Use the comma to help you decide what sentences to join when.

11. [No hint.]
15. [No hint.]
19. [No hint.]
23. You’ll need to use several connectives here.
25. Remember to distinguish what a sentence says from whether it is true.
30. See page 8 for examples of a production graph.
40. Remember that a disjunction is true iff at least one of the disjuncts is true.
42. Show this via a truth table.
45. Your first line might look like this: \((x = 0 \lor 1 - y^2 = 0) \land (x + 2 = 0 \lor y = 0)\). Keep breaking it down further, until you arrive at what the solution pair(s) \((x, y)\) must be, provided there are any.
1.3 Argument Semantics for SL

In Section 1.1 we noted that the validity of a given argument is independent of the actual truth or falsehood of the particular sentences involved, but that it depends upon all possible truth-value combinations for premises and conclusions having the same logical forms.

In this section we will focus on validity and logical implication relative to Sentential Logic. We will begin, though, by discussing the closely related notion of logical equivalence. To conclude this section, we will introduce the concepts of consistency, independence, and completeness on account of their importance for mathematical foundations; but we will not go very deeply into these topics since our aim here is the art of proof making.

Logical Equivalence in SL

The concept of logical equivalence can be defined in terms of logical implication, but equivalence seems to be the less complex notion, so we will begin with that. Two distinct sentences are logically equivalent when they say the same thing in different ways, using different logical operators/logical forms. Thus, two sentences are logically equivalent when their logical forms have exactly the same truth value for any given truth-value assignment. When two sentences are logically equivalent, we will insert the symbol $\equiv$ between them:

$$P \equiv Q$$

is read “$P$ is logically equivalent to $Q$.”

**DEFINITION 1.3-1: Logical Equivalence**

$P \equiv Q$ iff $P$ and $Q$ have identical truth tables.

It is rather easy to find pairs of logically equivalent sentences. It should be intuitively clear, for instance, that conjunction and disjunction are commutative operators, in the sense that $P \land Q \equiv Q \land P$ and $P \lor Q \equiv Q \lor P$. This can be shown with a simple truth table (Exercise 3).

Naturally, there are more interesting examples of logical equivalents than these. The Law of Double Negation (DN) is one of them. This asserts that negation is a toggle operator, satisfying $\neg\neg P \equiv P$ (Exercise 2). Two other important (but less obvious) equivalences are known as De Morgan’s Laws (DeM) in honor of Augustus De Morgan, though they were known much earlier to the medieval philosopher William of Occam. They exhibit expanded equivalents for negated conjunctions and disjunctions:

$$\neg(P \land Q) \equiv \neg P \lor \neg Q$$

and its dual

$$\neg(P \lor Q) \equiv \neg P \land \neg Q.$$ 

We will show the first and leave the other one, which is similar, as an exercise (Exercise 9a).

**EXAMPLE 1.3-1**

Show $\neg(P \land Q) \equiv \neg P \lor \neg Q$, De Morgan’s Law for negating conjunction.

**Solution**

The following double truth table establishes the equivalence. Compare the columns below the main connectives (above the double underline).

<table>
<thead>
<tr>
<th>$\neg(P \land Q)$</th>
<th>$\neg P \lor \neg Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
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<tr>
<td>$T$</td>
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<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

Since these two truth-value assignments are exactly identical, the formulas are logically equivalent. Both sentences say the same thing, only in different words. One form uses the language of conjunction and negation, while the other uses negations and disjunction. Both

---

* This symbol ought to be standard, given the one for implication, but I don’t know of any logic texts that use it.
forms are alike true or false. In words, if it is not the case that both \( P \) and \( Q \), then either \( P \) is not the case or else \( Q \) is not the case; and conversely.

**Logical Implication in SL**

In Section 1.1 we said that a set of sentences \( P \) logically implies a sentence \( Q \) iff any transformation or re-interpretation of the non-logical terms that makes all the premises \( P \) true also makes the conclusion \( Q \) true.

Since SL deals with the whole sentence as its smallest unit, we can reformulate this definition strictly in terms of the truth values of the sentences involved without worrying about having to interpret the non–logical terms *within* the sentences. By letting the atomic sentences take on all possible truth-value combinations, we can be certain we have considered all cases that might actually arise for sentences of a given form, regardless of the meaning involved.

**DEFINITION 1.3 - 2: Logical Implication, Logical Consequence**

a) A set of sentences \( P \) logically implies a sentence \( Q \), written \( P \models Q \), iff *every* truth-value assignment making \( P \) true also makes \( Q \) true.

b) \( Q \) is a logical consequence of \( P \) iff \( P \models Q \).

We will list the individual sentences in the premise set \( P \) to the left of the double turnstile \( \models \), separated by commas, and put the conclusion to the right.

As the notation seems to suggest, logical implication is half of logical equivalence, at least when pairs of sentences are involved. If \( P \models Q \) and \( Q \models P \), then \( P \models Q \) for \( Q \) is true whenever \( P \) is, and \( P \) is true whenever \( Q \) is. The notion of logical implication goes beyond relating sentences in pairs, however. Several premises may be involved in generating a given consequence.

Both \( P \) and \( Q \) may be thought of as conditions in the context of logical implication. For suppose the sentence \( P \models Q \) is true. \( P \) is thus a *sufficient condition* for \( Q \); its being the case is sufficient to guarantee that \( Q \) is also the case. \( Q \) is often called a *necessary condition* for \( P \); for if \( P \) is the case and \( P \models Q \), then \( Q \) is necessarily so as well. Whether or not you think \( Q \) is a ‘condition’ in the strict sense, using this term for the consequent as well as the antecedent gives us a common name for referring to both components.

Truth tables provide us with a mechanical method or an *effective decision procedure* for testing logical implication relative to SL. We make an *extended truth table* containing columns for each of the formulas involved, separating the premises from the conclusion by a double vertical line, and then merely check whether the conclusion is true whenever the premises are jointly true. If this is the case, the conclusion follows from the premises; otherwise it does not – the argument is invalid so far as SL is concerned.

**EXAMPLE 1.3 - 2**

Show that *conjunction* implies *disjunction*, but not conversely;

i.e., show that \( P \land Q \models P \lor Q \), but \( P \lor Q \not\models P \land Q \). Thus, \( P \land Q \not\models P \lor Q \).

**Solution**

This is shown by the following extended truth table.

<table>
<thead>
<tr>
<th>( P \land Q )</th>
<th>( P \lor Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T ) ( T )</td>
<td>( T )</td>
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<tr>
<td>( T ) ( F )</td>
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<tr>
<td>( F ) ( F )</td>
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</tbody>
</table>
By comparing the truth values of the conclusion with those of the premise, we see that whenever a conjunction is true (row one), the associated disjunction is also true. The converse, however, is not the case. There is a truth-value assignment in which the disjunction is true while the conjunction is false (in fact, there are two of them: see the second and third rows). Thus, a disjunction does not imply the associated conjunction, and so the two sentence forms are not logically equivalent, either. A conjunction is \textit{logically stronger} than a disjunction. More is being asserted by the conjunction than the disjunction.

Arguments having more than one premise are treated similarly, except that each premise is given a separate column preceding the double line.

\boldsymbol{\textsc{Example 1.3-3}}

Show that by ruling out one alternative in a disjunction, we may conclude the remaining disjunct; i.e., show that $P \lor Q, \neg P \models Q$. This result lies behind the basic strategy for playing Sudoku.

\textbf{Solution}

This is shown by means of the following extended truth table. The row that establishes the implication is the third one: when both premises are true, the conclusion is also true.

\begin{center}
\begin{tabular}{ccc}
$P \lor Q$ & $\neg P$ & $Q$ \\
$T$ & $T$ & $T$ \\
$T$ & $F$ & $F$ \\
$F$ & $T$ & $T$ \\
$F$ & $F$ & $F$
\end{tabular}
\end{center}

Full truth tables are useful but aren’t really necessary in order to show $P \models Q$. You merely have to determine what the truth of the premises $P$ dictates concerning the truth values of its constituent sentences and then show that $Q$ must also be true under those assignments. Thus, to reconsider the last example, if $\neg P$ is true, $P$ must be false; but then $Q$ must be true to make $P \lor Q$ true. So whenever the premises are both true, the conclusion is, too. To show $P \not\models Q$ is done in the opposite way: you must find an assignment that makes $Q$ false while keeping the premises $P$ true. Alternatively, you can give a counterargument to show that a given sentence does not follow from another set of sentences: formulate a concrete argument in the same form in which the premises are true but the conclusion is false.

Reading sentences such as $P \land Q \models P \lor Q$ shouldn’t present any difficulty. The symbols $\models$ and $\vdash$ are relation symbols, not logical operators. They are thus similar to $\leq$ and $=$ in algebra. Everything left of the symbol $\models$ implies/is equivalent to everything right of the symbol. The sentence $P \land Q \models P \lor Q$ is thus not a compound SL-sentence; it is a claim about how SL-sentences $P \land Q$ and $P \lor Q$ are related. It belongs to what is called the \textit{meta-theory} of SL. Such claims are similar to ones like $3 \leq 3 + 1$, which make claims about numbers and are not themselves numbers. We will return to explore this distinction further in the next section.

The notation of logical implication is also useful for exhibiting a sentence as a tautology: you indicate that the sentence follows without assuming any premises, placing it after the double turnstile. The assertions $\models \neg(P \land \neg P)$ and $\models P \lor \neg P$ say that these instances of the \textit{Law of Non-Contradiction} and the \textit{Law of Excluded Middle} are logical truths.

\textbf{Logical Implication and Equivalence Relative to Mathematics}

In developing a deductive theory for some branch or deductive theory of mathematics, mathematicians are usually concerned not with pure logical implication or logical equivalence,
but with these concepts relative to the theory under discussion. For instance, when mathematicians claim that a result is logically equivalent to Euclid’s Parallel Postulate, they usually mean this in the context of the rest of Euclidean Geometry, not in an absolute sense. The same thing is true in Linear Algebra when a string of statements is claimed to be equivalent to an associated matrix being invertible, or in Peano Arithmetic (see Section 3.3) when the Well-Ordering Principle is said to be equivalent to the Principle of Mathematical Induction.

We can define these relativized notions as follows. Think of $A$ as a set of axioms for a mathematical theory and the deductive theory of $A$ as all the logical consequences of $A$.

**Definition 1.3-3: Logical Implication and Equivalence Relative to a Theory**

a) A set of sentences $P$ logically implies a sentence $Q$ relative to $A$, symbolized by $P \models_A Q$, iff $A, P \models Q$.

b) A sentence $P$ is logically equivalent to a sentence $Q$ relative to $A$, symbolized by $P \equiv_A Q$, iff $P \models_A Q$ and $Q \models_A P$.

This relativized notion of logical implication will prove useful in Section 1.4 when we discuss the Implication Theorem. It will help us explain how certain misperceptions and abuses of terminology connected to the idea of logical implication arise in mathematical circles.

**Consistency, Independence, and Completeness for SL**

When mathematicians axiomatize a given theory, they use a number of criteria to guide them in their work. They want first of all to choose a foundation consisting of results that are basic to the field and that are relatively simple. Judging propositions in this regard is a matter of trained mathematical intuition and (partly) individual taste, not logic. However, mathematicians also have a number of logical concerns to keep in mind. They certainly want their axioms to be consistent with one another; they may want them to be independent of one another to avoid unnecessary overlap; and they will want them to provide a deductive basis adequate to prove all known results, insofar as that is possible. We will look at each of these logical notions in turn: consistency, independence, and completeness.

A collection of sentences is logically consistent iff the sentences do not contradict one another; that is, iff some of them being true does not force others to be false. Note carefully: the issue here is one of necessary logical connection, not actual truth values of any particular sentences involved. What is important, therefore, are the logical forms involved and their possible interpretations and truth values. This leads us to the following definition.

**Definition 1.3-4: Consistent and Inconsistent Sentences**

a) A set of sentences $P$ is consistent iff there is some truth-value assignment that makes all the sentences in $P$ true.

b) A set of sentences $P$ is inconsistent iff it is not consistent; i.e., iff there is no truth-value assignment that simultaneously assigns the value $T$ to all sentences in $P$.

To illustrate these concepts, it can be shown that $P \lor Q, \neg Q \lor R$, and $\neg P \land R$ are consistent, while $P \lor Q, \neg Q \lor R$, and $\neg P \land \neg R$ are inconsistent (see Exercise 29a).

Mathematicians and even logicians occasionally discover to their chagrin that sets of sentences they thought were consistent are not so after all. If this occurs for the axioms of some deductive theory, it devastates the entire system from a logical point of view. For not only does this call into question the truth of the conclusions already obtained, it thus also becomes theoretically possible, as we will see later, to prove any proposition whatsoever formulated in the language of that theory. This makes an inconsistent set of axioms worthless for establishing a theory. Consistency is thus an absolute prerequisite for the development of an axiomatic theory. For this reason, demonstrating consistency has been a primary goal of mathematical logicians ever since foundational issues became a central concern about a century ago.
Another prerequisite that can be placed upon an axiomatic theory is that its axioms be *logically independent*. This means roughly that the sentences have no logical connection to one another.

**DEFINITION 1.3 - 5: Independent Sentences**

A sentence $Q$ is independent of a set of sentences $\mathcal{P}$ iff neither $Q$ nor $\neg Q$ is logically implied by $\mathcal{P}$: i.e., $\mathcal{P} \not\models Q$ and $\mathcal{P} \not\models \neg Q$.

A sentence $Q$ is therefore independent of $\mathcal{P}$ iff one truth-value assignment makes $Q$ and all the sentences of $\mathcal{P}$ true and another one makes the sentences of $\mathcal{P}$ true but $Q$ false. Using the definition of consistency, it can easily be shown that $Q$ is independent of $\mathcal{P}$ iff each of the two sets $\{\mathcal{P}, Q\}$ and $\{\mathcal{P}, \neg Q\}$ is logically consistent (see Exercise 29c).

The requirement that a set of axioms be independent is largely one of economy: don’t accept as axiomatic what you can prove as a theorem. Pushing this to the limit can be counterproductive, though. Adopting a set of completely independent axioms often makes the deductions of the system unbearably long and difficult. At other times, however, independence is a major concern. Mathematicians often want to know, for instance, whether either a given result or its negation are implied by the axioms of a theory. If neither are, then it must be decided whether or not to adopt it as a new axiom in an extension of the original theory. This was the case historically in geometry. Mathematicians discovered in the nineteenth century after more than 2000 years of unsuccessful efforts to prove Euclid’s Parallel Postulate that it was independent of the other axioms of geometry. Some of the most important results in set theory and logic of the past century have also been independence results. These results go far beyond SL, however, so we will discuss them here.

Finally, closely related to the notions of consistency and independence is the concept of theory completeness. A theory is *complete* iff no sentence in the language of the theory is independent of its axioms; equivalently iff given any sentence in the language, either it or its negation logically follows from the axioms of the theory.

**DEFINITION 1.3 - 6: Theory Completeness**

A theory axiomatized by $\mathcal{A}$ is complete iff given any sentence $Q$ in the language of $\mathcal{A}$ either $\mathcal{A} \models Q$ or $\mathcal{A} \models \neg Q$.

Thus, any genuine extension of a complete theory is necessarily inconsistent, since it would have as consequences both a sentence and its negation.

Sometimes mathematicians aim at developing a complete theory, and sometimes they do not. For instance, it would be good to have a complete theory of ordinary arithmetic or Euclidean geometry, since that is intended to be a theory about a specific structure (Euclidean space, the system of natural numbers). But the theory of vector spaces and group theory, on the other hand, are incomplete by design: they’re supposed to provide a general theory for mathematical structures having very different properties, so the axiomatic basis for these theories will not be able to prove all possible results about such structures.

You might wonder whether axiomatizations of complete mathematical theories even exist. Without going into details, the answer is yes. However, there are also well known incompleteness results due to Gödel and others that apply even to some fairly simple mathematical theories, such as axiomatized arithmetic. Once again, this goes beyond SL and beyond what we will be doing in this book. Treatment of this topic and the more fundamental topics of consistency and independence can be found in an advanced book on mathematical logic.
EXERCISE SET 1.3

Problems 1-7: Logical Equivalences
Show that the following equivalences hold.

1. Idempotence Laws
   a. \( P \land P \vdash P \)
   b. \( P \lor P \vdash P \)

2. Law of Double Negation
   \( \neg \neg P \vdash P \)

3. Commutative Laws
   a. \( P \land Q \vdash Q \land P \)
   b. \( P \lor Q \vdash Q \lor P \)

4. Associative Laws
   a. \( (P \land Q) \land R \vdash P \land (Q \land R) \)
   b. \( (P \lor Q) \lor R \vdash P \lor (Q \lor R) \)

*5. Absorption Laws
   a. \( P \land (P \lor Q) \vdash P \)
   b. \( P \lor (P \land Q) \vdash P \)
   c. \( P \land (Q \lor \neg Q) \vdash P \)
   d. \( P \lor (Q \land \neg Q) \vdash P \)

*6. Distributive Laws
   Show that conjunction distributes over disjunction, much like multiplication distributes over addition in arithmetic and algebra.
   a. \( P \land (Q \lor R) \vdash (P \land Q) \lor (P \land R) \)
   b. \( P \lor (P \land Q) \vdash (P \lor Q) \land (P \lor R) \)

*7. More Distributive Laws
   Ordinary addition does not distribute over multiplication in arithmetic: \( 3 + 5 \times 2 \neq (3 + 5) \times (3 + 2) \). Does disjunction distribute over conjunction in Sentential Logic? Determine the truth of the following.
   a. \( P \lor (Q \land R) \vdash (P \lor Q) \land (P \lor R) \)
   b. \( P \land (Q \lor R) \vdash (P \land Q) \lor (P \land R) \)

8. Find a simpler logical equivalent to the following clause, taken from a computer program: \( \text{while } ((x < 40 \ \text{AND} \ y > 90) \ \text{OR} \ (x < 40 \ \text{AND} \ (y > 90 \ \text{OR} \ z > 10))) \), do . . . . You may want to use letters to symbolize the atomic sentences to help you determine an equivalent. Check your answer with a truth table.

*9. De Morgan’s Law for Negating a Disjunction
   a. Show the dual law to the one given in Example 1.3-1: \( \neg (P \lor Q) \vdash \neg P \land \neg Q \).
   b. Determine an equivalent to \( \neg (P \land \neg P) \). Using the Law of Double Negation, simplify your answer. What laws are these final sentences instances of?
   c. Carefully interpret the meaning of the mathematical symbolism \( x \neq \pm 1 \) by putting it into expanded form, using logical symbolism for all the connectives involved. Then state an equivalent sentence using De Morgan’s Law.

*10. Show that logical equivalence is an equivalence relation. That is, argue that \( \vdash \) satisfies the following reflexive, symmetric, and transitive properties:
   i. Reflexive: \( P \vdash P \)
   ii. Symmetric: if \( P \vdash Q \), then \( Q \vdash P \)
   *iii. Transitive: if \( P \vdash Q \) and \( Q \vdash R \), then \( P \vdash R \)
11. Is the relation of logical implication an equivalence relation? That is, does $\models$ satisfy all of the following properties? If not, which ones hold for $\models$?
   i. Reflexive: $P \models P$
   ii. Symmetric: if $P \models Q$, then $Q \models P$
   iii. Transitive: if $P \models Q$ and $Q \models R$, then $P \models R$.

**Problems 12-16: Logical Implications**

Show that the following implications hold.

12. **Simplification**
   a. $P \land Q \models P$
   b. $P \land Q \models Q$

13. **Addition**
   a. $P \models P \lor Q$
   b. $Q \models P \lor Q$

*14. $\neg(P \land Q), P \models \neg Q$

*15. $(P \lor Q) \land R, \neg P \models Q \land R$

16. $P \lor Q, \neg P \land \neg Q \models R$

*17. Are the following expansion and contraction laws for conjunction legitimate? Explain. You do not need to write out a full truth table to decide.
   a. Expansion: $P \land Q \models (P \lor R) \land (Q \lor S)$
   b. Contraction: $(P \lor R) \land (Q \lor S) \models P \land Q$

**Problems 18-21: True or False**

Are the following statements true or false? Explain your answer.

*18. The symbol ‘$\models$’ represents a truth-functional connective.

*19. A mathematical result is independent of a set of statements iff it cannot be proved from them.

20. Mathematicians want their axiom systems to be logically consistent.

21. Many axiom systems in mathematics are incomplete by design.

**Problems 22-25: Defining Terms**

Give the meaning of the following terms, putting it into your own words.

22. Logically equivalent statements

*23. Logical consequence of a set of premises

24. A consistent set of statements

25. Theory completeness

**Problems 26-29: Consistency, Implication, and Independence**

The following problems explore the meanings of and connections between consistency, implication, and independence.

*26. Logical Falsehoods, Inconsistent Sentences, and Logical Consequences
   a. Explain why a logical falsehood implies any sentence whatsoever.
   b. Sometimes mathematicians make a stronger claim than a; namely, that a false sentence implies any sentence. Is this correct? Explain your answer carefully.
   c. Does an inconsistent set of sentences imply any sentence whatsoever? Why or why not?
27. **Contradictory Sentences, Inconsistent Sentences**  
*Contradictory sentences* are sentences that have opposite truth values for all truth-value assignments.

a. Show that $P \land \neg Q$ and $\neg P \lor Q$ are contradictory.
b. Give a brief argument to show that if a sentence $P$ contradicts $Q$, then the set $\{P, Q\}$ is inconsistent.
c. Is the converse to part b also true? That is, if the set $\{P, Q\}$ is inconsistent, must $P$ and $Q$ contradict one another? Prove it if it is true; give a counterexample if it is false.

*28. Consistency and Implication***

a. If a sentence $Q$ follows from a set of consistent sentences $\mathcal{P}$, does $Q$ have to be consistent with $\mathcal{P}$? Give reasons or a counterexample to support your answer.
b. If $Q$ is consistent with $\mathcal{P}$, does $Q$ have to follow from $\mathcal{P}$? Give reasons or a counterexample to support your answer.

29. **Independence**

a. Show that the sentences $P \lor Q$, $\neg Q \lor R$, and $\neg P \land R$ form a consistent set of sentences, but that the sentences $P \lor Q$, $\neg Q \lor R$, and $\neg P \land \neg R$ are inconsistent.
b. Show that $\neg P \land R$ is independent of the sentences $P \lor Q$ and $\neg Q \lor R$. Are either $P \lor Q$ or $\neg Q \lor R$ independent of the other two sentences?
c. Using the definition of independence, show that a sentence $Q$ is independent of a set of sentences $\mathcal{P}$ iff each of the two sets gotten by separately adjoining $Q$ and $\neg Q$ to $\mathcal{P}$ is logically consistent.

**Problems 30 - 32: Other Conjunctive and Disjunctive Connectives**

The following problems explore additional connectives related to $\land$ and $\lor$.

**EC 30. Exclusive and Non–Exclusive Disjunction**

a. Write out the truth table for $P \nand Q$, where $\nand$ denotes exclusive disjunction.
b. Show that $\nand$ can be “defined” in terms of $\land$, $\lor$, and $\neg$; that is, find a sentence involving these connectives and the sentence symbols $P$ and $Q$ that is logically equivalent to $P \nand Q$.
c. Show how $\nand$ can be defined in terms of $\land$ and $\lor$; that is, find a sentence involving these connectives and the sentence symbols $P$ and $Q$ that is logically equivalent to $P \lor Q$.
d. True or False: $\neg(P \nand Q) \vdash \neg P \land \neg Q$.

**EC 31. Let $P \triangledown Q$ (read: $P$ NOR $Q$) be an abbreviation standing for negated conjunction (not–and): $\neg(P \land Q)$.

(Think of the bottom line of $\triangledown$ as negating the $\land$ sign on top of it.) This connective is often symbolized by $|$ and is called the Sheffer stroke after H. M. Sheffer, who investigated its properties in 1913.

a. Write a truth table for $P \triangledown Q$.
b. Show that $\neg P \vdash P \triangledown P$.
c. Show that $P \land Q \vdash (P \triangledown Q) \triangle (P \triangledown Q)$.
d. Determine a logical equivalent involving $\triangle$ for $P \lor Q$.
e. Exploration: what sorts of laws (commutative, associative, distributive, absorption, etc.) hold for the connective $\triangle$?

**EC 32. Let $P \triangledown Q$ (read: $P$ NOR $Q$) be an abbreviation standing for neither $P$ nor $Q$; i.e., for $\neg(P \lor Q)$.

(Think of the top line of $\triangledown$ as negating the $\lor$ sign below it.) This connective is often symbolized by the down arrow $\triangledown$, which can also be thought of as a slashed or negated ‘or’ sign.

a. Write a truth table for $P \triangledown Q$.
b. Show that $\neg P \vdash P \triangledown P$.
c. Show that $P \lor Q \vdash (P \triangledown Q) \triangledown (P \triangledown Q)$.
d. Determine a logical equivalent involving $\triangledown$ for $P \land Q$.
e. Exploration: what sorts of laws (commutative, associative, distributive, absorption, etc.) hold for the connective $\triangledown$?

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* Unfortunately, Tarski used $\triangledown$ to denote neither–nor, our $\triangledown$, in the widely circulated textbook, *Introduction to Logic and to the Methodology of Deductive Sciences*, which makes the symbolism we have adopted somewhat less desirable.
33. *Logically Equivalent Sentences and Logical Implication*

a. Show that two sentences \( P \) and \( Q \) are logically equivalent iff their sets of implied sentences \( R \) are identical; i.e., \( P \vdash Q \) iff \( P \models R \) whenever \( Q \models R \) and conversely.

b. We will call two sets of sentences \( P \) and \( Q \) logically equivalent when each sentence of \( P \) is logically implied by \( Q \) and each sentence of \( Q \) is implied by \( P \). Using this definition, generalize the result given in part a to cover the case when sets of sentences are involved. What does this result say about axiomatizing a mathematical theory by means of a set of sentences that are equivalent to a given set of axioms?

c. Show that the extended notion of logical equivalence defined in part b is an equivalence relation (see Exercise 10 above).
HINTS TO STARRED EXERCISES 1.3

5. a. Show this with a truth table.
6. a. Show this with a truth table.
7. a. Make an extended truth table, then determine whether the first statement is true if and only if the second statement is as well.
9. a. Show this by means of a truth table.
   c. The expanded basic form of $x \neq \pm 1$ should be two negated atomic sentences joined with a conjunction.
10. iii. Remember that $P \vDash Q$ means that $P$ and $Q$ have exactly the same truth tables.
14. To show logical implication, you must show that when all the premises are true, so is the conclusion.
15. To show logical implication, you must show that when all the premises are true, so is the conclusion.
17. a. Remember that $A \models A \lor B$.
   b. Remember that $A \lor B \nvDash A$.
18. [No hint.]
19. [No hint.]
23. [No hint.]
26. a. For logical implication to hold, the conclusion only needs to be true whenever all the premises are true.
   b. A sentence that is false, but not logically false, must have an interpretation that makes the sentence true.
   c. For logical implication to hold, the conclusion only needs to be true whenever all the premises are true.
28. a. If $\mathcal{P} \models Q$, then whenever $\mathcal{P}$ is true, so is $Q$.
   b. For a sentence $Q$ to be consistent with a given set of sentences $\mathcal{P}$, there only needs to be some truth value assignment making all the sentences true.
1.4 Conditional and Biconditional Sentences

Besides “and”, “or”, and “not”, there are two other very important sentential connectives: the conditional connective “if - then” and the biconditional connective “if - and - only - if” (“iff”). The latter connective is composed of “if - then” and “and”, so we will treat “if - then” first and in more detail.

A compound sentence of the form ‘if \( P \) then \( Q \)’ is a \textit{conditional sentence}. \( P \) is called the \textit{antecedent} of the sentence, and \( Q \) is the \textit{consequent}. There are several notations in use for the “if - then” connective: the sideways horseshoe \( \supset \) is often used in logic textbooks written for philosophy students; a double-shafted arrow \( \Rightarrow \) is popular among mathematicians; and a single-shafted arrow \( \rightarrow \) is used by mathematicians and logicians alike. We will always symbolize “if - then” by means of the simple arrow, writing ‘if \( P \) then \( Q \)’ as \( P \rightarrow Q \). The biconditional sentence ‘\( P \) iff \( Q \)’ will be symbolized by \( P \leftrightarrow Q \).

\textbf{Plan for Treating Conditional Sentences}

There are several ways to motivate the conventional truth table for conditional sentences. The most popular procedure is to take a specific conditional sentence and use it as a prototype for determining the truth table for all such sentences. The truth table is then further verified by observing that conditional sentences should be taken as logically equivalent to certain sentences involving \( \neg \), \( \land \), and \( \lor \) and that these sentences generate the same table of values.

This approach has the advantage of getting one to accept the conventional truth table with a minimum of fuss. It is also too slick. It glosses over rather than explains some of the real difficulties people have with the standard truth-value assignment. So while we will present this approach first, we will supplement it by another one that goes deeper into the whole matter.

Our second approach will explore the possible connection between the “if - then” \textit{operator} inside SL on the sentence level and the “if - then” \textit{relation} of logical implication on the argument level. This approach is both more natural and more honest than the first one, since a connection between implication and the connective “if - then” has always been recognized, though the precise relationship has often been misconstrued. Clarifying the issue may involve some hard thinking on your part (it’s a topic that even trips up many mathematicians), but in the end you should have a much better understanding of why the standard truth-functional definition of “if \( P \) then \( Q \)” is exactly what we want. As a reward for your patience and hard work, the main result relating conditional sentences to logical implication (the \textit{Implication Theorem}) will come as no big surprise to you, being the underlying motivation all along.

\textbf{The Truth Table for IF-THEN: A Simple Motivation}

Consider the following conditional sentence from elementary geometry about quadrilaterals:

If the sides of \( A'B'C'D' \) are congruent to the corresponding sides of \( ABCD \), then \( A'B'C'D' \) is congruent to \( ABCD \).

We can denote this sentence by \( P \rightarrow Q \), where \( P \) stands for ‘the sides of \( A'B'C'D' \) are congruent to the corresponding sides of \( ABCD \)’ and \( Q \) represents ‘\( A'B'C'D' \) is congruent to \( ABCD \)’.

How could you tell whether this sentence is true or false? Well, you could examine many different pairs of quadrilaterals \( ABCD \) and \( A'B'C'D' \) that match up side for side and see whether they are fully congruent. If you can find a pair of quadrilaterals that aren’t, then you know the assertion is false: you have a counterexample. But if you can show that no such pair of non-congruent quadrilaterals exists, then the statement must be true.* Generalizing from

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* Whether the sentence is actually true or false is of no concern to us at the moment. See Exercise 19, however.
this, we can say that $P \rightarrow Q$ is false if $P$ is true while $Q$ is false; but otherwise $P \rightarrow Q$ is true. We thus have the standard truth-functional assignment for conditional sentences: $P \rightarrow Q$ is true unless it is clearly false, the logical version of “innocent until proven guilty”.

**DEFINITION 1.4-1: Truth-Value Assignment for Conditional Sentences**

The conditional sentence $P \rightarrow Q$ is false if $P$ is $T$ and $Q$ is $F$; else it is true.

In positive terms: $P \rightarrow Q$ is true iff $P$ is $F$ or $Q$ is $T$.

This yields the following truth table for $P \rightarrow Q$.

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>$P \rightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

To support this truth-value assignment, we can appeal to the intended meaning of the sentence $P \rightarrow Q$. Doesn’t this sentence mean that whenever $P$ is the case, $Q$ must be, too? In other words, assuming that $P \rightarrow Q$ is the case, it cannot happen that $P$ is true and $Q$ is false, and conversely. Translating this assertion into one about sentences of SL, $P \rightarrow Q$ must be logically equivalent to $\neg(P \land \neg Q)$. Alternatively, asserting $P \rightarrow Q$ is tantamount to saying either $P$ is not the case or else $Q$ is. This gives us the equivalent $\neg P \lor Q$. Since these two alternatives are themselves equivalent (simplify the former by means of De Morgan’s Law and Double Negation; or simply write out truth tables for both – see Exercise 73), and since they have the same truth table as what is given above, this surely gives further evidence for the correctness of our truth-value assignment.

**The Truth Table for IF-THEN: A Closer Look**

But now it’s time to inject a measure of skepticism about the conventional truth-value assignment. This will help us gain a deeper appreciation for the nature of the conditional connective. Look at the truth table for $P \rightarrow Q$ a little more closely, line by line. The second row is completely unproblematic: if $P$ is true while $Q$ is false, the sentence $P \rightarrow Q$ cannot possibly be true given how “if-then” is ordinarily used. The truth-value assignments for the other three rows, however, do not fare as well, as we will now argue.

In the case where both $P$ and $Q$ are true (row one), the value $T$ is assigned to $P \rightarrow Q$. Should this be the case? We can certainly find specific sentences that make this seem reasonable. Consider the simple sentence of number theory

\[ \text{If } a \text{ is even, then } a^2 \text{ is even}. \]

Now suppose $a = 2$, so that $a^2 = 4$. Both of these numbers are even, so the antecedent and consequent are both true. And the conditional itself is true: even times even is even. So if the truth value of a conditional sentence is uniquely determined by the truth values of its components sentences, it seems that $P \rightarrow Q$ must be true when both $P$ and $Q$ are true.

On the other hand, accepting this truth-value assignment means some conditionals will have to be considered true even though no logical connection holds between the two parts. The sentence

\[ \text{If } 2 \text{ is even, then } 17 \text{ is prime} \]

probably deserves a blank look, but it must be taken as true according to our assignment scheme. Maybe you think this sentence (but not the last) should be considered false, since “17 is prime” doesn’t seem to follow from “2 is even”. But this would mean abandoning the

1.4-2
idea that “if-then” is a simple truth-functional connective, for then the truth value of $P \rightarrow Q$ would depend on more than the truth values of the constituent sentences involved.

The truth values of the third and fourth lines of the truth table for $P \rightarrow Q$ are also $T$, and for the same basic reason as row one. For example, the conditional sentence

$$\text{If } ABCD \text{ is a square, then } ABCD \text{ is a rectangle}$$

is certainly true, given the accepted meaning of ‘square’ and ‘rectangle’. It remains true even if a particular figure $ABCD$ happens to be a non-square rectangle, making the antecedent false and the consequent true. It is also true even if the figure $ABCD$ is a non-rectangular quadrilateral, so that both antecedent and consequent are false. Thus, it does seem reasonable to enter $T$ in lines three and four of the truth table. There is no other way to consider the above statement true and have the truth of the conditional depend solely upon the truth values of the component sentences.

Once again, though, we are forced to accept some rather strange sentences as being true under these assignments. For instance, the sentences

$$\text{If } 0 = 1, \text{ then } \sqrt{2} \text{ is irrational}$$

and

$$\text{If all triangles are equilateral, then all circles are squares}$$

must both be considered true statements according to the standard truth-value assignment, even though in neither case does the first clause imply the last one. With a little imagination, you can come up with even more graphic examples, especially if you use non-mathematical sentences.

The strangeness involved in these three cases occurs because the component sentences being joined by “if-then” have no intimate logical relation to one another. You would never be tempted to make such weird statements in a mathematics class (except when talking logic!), but if $\rightarrow$ is to be a bona-fide truth-functional sentential connective or logical operator, you must be able to use it to connect any two sentences whatsoever and then uniformly assign a truth value to the result based solely upon the truth values of the constituents. We are thus faced with a dilemma: do we accept the truth values we have stipulated above ($T$), or do we change them (to $F$), since that may be more appropriate for the silly examples?

It should be clear from the above remarks that the truth table for conditional sentences was initially filled out on the basis of a fairly specialized sort of example and that the values chosen are in some sense conventional. The conditionals we chose to confirm the correctness of the truth table were ones in which the first sentence logically implied the second; the problem conditionals that we chose to ignore were ones in which this did not occur.

Our discussion of the meaning of $P \rightarrow Q$, both initially and later in terms of equivalent sentences involving $\neg$, $\wedge$, and $\vee$, was also predicated on a tacit understanding of the conditional sentence as asserting that $Q$ follows from $P$. This being the case, shouldn’t the precise connection between conditional sentences and assertions of logical implication be made more explicit, so that the two concepts can be properly distinguished and related? We will now proceed to do just that. This will also provide us with a second, deeper and more genuine justification for the conventional truth table of $P \rightarrow Q$.

**Conditional Sentences vs. Logical Implication**

You may think (as many do) that the “if-then” connective very nicely captures the relationship of logical implication, and that you can simply read a sentence $P \rightarrow Q$ as $P$ implies $Q$; in other words, that $\rightarrow$ amounts to $\vdash$. Identifying the conditional connective with the logical relation of implication, however, is wrong, for several important reasons.
First of all, while → has been introduced as a *truth-functional connective*, ⊨ is no such thing. The truth or falsehood of the meta-logical assertion \( P \models Q \) is largely independent of the truth values of the component sentences. We learned this already in Section 1.1. Except for when \( P \) is \( T \) and \( Q \) is \( F \), which makes \( P \models Q \) false, \( P \models Q \) can be either true or false for a given truth-value assignment; it all depends on the logical connection holding between \( P \) and \( Q \). Thus \( \models \) is not truth-functionally determined and so cannot be the same as \( \rightarrow \).

Secondly, and just as importantly, identifying \( \rightarrow \) and \( \models \) wrongly confuses different levels or domains of language in an unacceptable way. \( P \rightarrow Q \) is a compound sentence in the object language of SL: it is formed by applying the conditional operator to \( P \) and \( Q \), and it makes a conditional assertion about the subject matter mentioned by \( P \) and \( Q \), such as quadrilaterals. \( P \models Q \) is a very different sort of sentence. It is an assertion about how the sentences \( P \) and \( Q \) are logically related, not a compound sentence about quadrilaterals. The one sentence is a part of mathematics; while the other is a part of logic. The first belongs to the object language, the second to the meta-language. As sentences in very different domains of discourse, belonging to different levels of language, they should not be identified.

To clarify this further, we might think of SL as an algebraic structure equipped with certain operations, functions, properties, and relations. The objects of this mathematical system are the well-formed formulas or sentences of SL, while sentential connectives like \( \neg \), \( \land \), and \( \lor \) that operate upon these objects to yield other compound formulas are unary and binary *operations* of the system. Truth-value assignments are *functions* defined on the set of formulas taking on the (truth) values \( T \) and \( F \). The system SL also has certain logical *properties* and *relationships* that hold for and between various elements. Some SL sentences have the property of being logically true or false, while others are logically indeterminate. Logical implication and logical equivalence are relationships holding between objects of this system. Although logical implication is defined more broadly than for pairs of sentences, for comparison purposes we will restrict our attention to that case, treating it as a binary relation.

In identifying \( \rightarrow \) with \( \models \), a binary *operator* is being identified with a binary *relation*, something that strictly speaking makes no sense. No one would ever dream of identifying a computational operation, such as subtraction, with an arithmetic relation, such as “is less than”.* Subtraction operates on numbers and yields numbers. The relation “is less than” is used to make an assertion about numbers and so yields not a number but a statement.

In mathematical logic, however, confusing the operator \( \rightarrow \) with the relation \( \models \) is easier to do because the objects of the system under investigation are not numbers but are themselves sentences, as are the assertions about them. Furthermore, in expressing both the conditional connective and the relation of logical implication one often makes use of the language “if-then”, and this makes them seem identical. So the confusion is more easily generated in this case and requires a bit more sophistication to ferret out and guard against. But forewarned is forearmed, so you should now have a better chance of distinguishing the two concepts.

### Relating Conditional Sentences and Logical Implication

Having emphasized the difference between logical implication and the conditional connective, we will now turn around and focus on what links them together. We will discover that there is a rather intimate connection between them.

The linkage between \( \rightarrow \) and \( \models \) is perhaps best approached by way of analogy, following up on our algebraic viewpoint of the last subsection. In order to capture the notion of a particular relation, one might make use of the operations and properties defined for the algebraic system.

---

* Well, not never. As anyone who has taught students the meaning of the relation “divides” in number theory can attest, beginning students often want to replace this relation with the operation of division, so a similar sort of confusion does take place in some areas of mathematics.
Thus, to explain what \( a < b \) means, we can say it means that \( b - a \) is positive. Here the relation \( < \) is defined in terms of an operation \( - \) and a property of that operation result. This tack is taken in many other cases, too: it is actually a very familiar mathematical maneuver. To take a number theoretic example, the relation “divides” is defined in terms of the operation of division and the property of having no remainder: \( a \) divides \( b \) iff \( b \) divided by \( a \) leaves a remainder of \( 0 \). Can we do the same sort of thing for logical implication? Can we capture the meaning of the relation \( \models \) by using an operator/connective and a property of that operation result?

The obvious operator to try to relate \( \models \) to is \( \rightarrow \), given the similarity of expression involving ‘if-then’. But what property of sentences might we use? The only property readily available to us is that of being true or false. Can we say that \( P \models Q \) iff \( P \rightarrow Q \) is true?

This is close, but it’s not quite correct, as a moment’s reflection will convince you. For the assertion \( P \models Q \) holds or doesn’t hold based on the logical form of the sentences, not on the actual sentences involved or their particular truth values. If we are to invoke the same degree of generality for \( P \rightarrow Q \) as for \( P \models Q \), we would have to say instead that \( P \rightarrow Q \) is true regardless of what \( P \) or \( Q \) means or what truth values they have; i.e., that \( P \rightarrow Q \) is a logical truth or tautology.

This connection turns out to be exactly correct. It is important enough to label it a theorem of logic.* We will state it in two forms, one for pure logical implication and one for logical implication relative to a deductive theory.

**THEOREM 1.4 - 1: Implication Theorem**

\[ P \text{ implies } Q \text{ iff } P \rightarrow Q \text{ is a tautology}; \text{ in symbols, } P \models Q \text{ iff } \models P \rightarrow Q. \]

**Proof:**

For easy reference, we will repeat the truth table for \( P \rightarrow Q \).

<table>
<thead>
<tr>
<th>( P )</th>
<th>( Q )</th>
<th>( P \rightarrow Q )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( T )</td>
<td>( F )</td>
<td>( F )</td>
</tr>
<tr>
<td>( F )</td>
<td>( T )</td>
<td>( T )</td>
</tr>
<tr>
<td>( F )</td>
<td>( F )</td>
<td>( T )</td>
</tr>
</tbody>
</table>

First suppose that \( P \models Q \). We must show that the sentence \( P \rightarrow Q \) is true under all possible truth-value assignments compatible with this given.

Since \( P \models Q \), it is impossible for \( P \) to be true and \( Q \) to be false: row two just can’t happen for such a \( P \) and \( Q \).

Yet this is the only case in which \( P \rightarrow Q \) could possibly be false.

Thus, \( P \rightarrow Q \) is always true; i.e., \( \models P \rightarrow Q \).

Conversely, suppose \( P \rightarrow Q \) is always true, no matter what truth-value assignment is given to its atomic components.

Then \( P \) cannot be true while \( Q \) is false; we must again rule out line two of the truth table for such a \( P \rightarrow Q \).

Line one of the truth table is now all that is relevant for assessing whether \( P \) implies \( Q \). Thus, whenever \( P \) is true, \( Q \) is also true.

Hence, \( P \models Q \).  \( \blacksquare \)

Using a similar argument to the one just given, the Implication Theorem can be generalized to cover the case where a sentence \( Q \) follows from \( P \) in the context of a theory axiomatized by a set of axioms \( \mathcal{A} \).

---

* The associated deeper Deduction Theorems for a Hilbert-style deduction system was formulated and proved by Tarski in 1921; Herbrand came to it independently and published it in 1928. In its original context, this result played the role of Conditional Proof, a natural deduction inference rule we will introduce in Section 1.7.
THEOREM 1.4 - 2: Relativized Implication Theorem

\[ P \models_A Q \iff \models_A P \rightarrow Q; \text{ that is, } \models_A P \iff \models_A P \rightarrow Q. \]

Proof:

See Exercise 90.

We will illustrate the Implication Theorem with several examples. The first example will allow us to point out concretely both the difference and the similarity between stating a conditional sentence in the object language and asserting a relation of logical implication in the meta-language.

✠ EXAMPLE 1.4 - 1

Show that \((P \rightarrow Q) \lor (Q \rightarrow P)\) is a logical truth.

Solution

The truth table for \((P \rightarrow Q) \lor (Q \rightarrow P)\) is the following.

<table>
<thead>
<tr>
<th>((P \rightarrow Q) \lor (Q \rightarrow P))</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
</tr>
<tr>
<td>T</td>
</tr>
<tr>
<td>F</td>
</tr>
<tr>
<td>F</td>
</tr>
</tbody>
</table>

Even though \((P \rightarrow Q) \lor (Q \rightarrow P)\) is a tautology, it is not a conditional sentence \((\lor\) is the main connective), and the closest corresponding meta-language statement, \(P \models Q\) or \(Q \models P\), is quite obviously false. For, given two sentences \(P\) and \(Q\), neither one of them need logically imply the other one. The disjuncts \(P \rightarrow Q\) and \(Q \rightarrow P\) are both logically indeterminate, so by the Implication Theorem neither of the corresponding implications holds. The moral of this example is the same one we stated repeatedly above: you cannot just read \(P \rightarrow Q\) as \(P = Q\).

Well, we need to hedge this a little bit. Reading a conditional sentence \(P \rightarrow Q\) as a statement of logical implication (replacing \(\rightarrow\) by \(\models\)) does have limited value. According to the Implication Theorem, such a translation is one way to determine whether the conditional sentence \(P \rightarrow Q\) is a tautology. If the translation produces a true assertion about logical implication, then the original conditional is logically true. Conversely, if the original conditional is known to be a tautology, you are permitted to conclude that the associated assertion \(P \models Q\) is the case.

Actually, the Relativized Implication Theorem allows us to relax our caution even further. If a sentence \(Q\) follows from a sentence \(P\) relative to a given axiomatized theory, then the sentence \(P \rightarrow Q\) follows from the axioms of that theory. Conversely, if \(P \rightarrow Q\) is true in the context of a deductive theory, then \(P \models Q\) holds relative to that theory. Thus, while \(P \rightarrow Q\) doesn’t itself say that \(P\) implies \(Q\) (and still shouldn’t be read that way), its status as a theorem does allow you to say so relative to a deductive theory, thanks to the Relativized Implication Theorem. If \(P \rightarrow Q\) is a theorem of some deductive theory, \(P\) does imply \(Q\) relative to that theory. This comes very close, then, to justifying the mathematical custom of reading ‘if-then’ statements as logical implications. In general, however, to revert back to our main contention, it is not only misleading but wrong to blur the distinction involved and interpret a conditional sentence formulated in the language of the theory as making an assertion of logical implication.

The next two examples explore the association between logical implication and logical truth a bit further.

✠ EXAMPLE 1.4 - 2

Show that \((P \land Q) \rightarrow (P \lor Q)\) is a tautology.
Solution
We showed in Example 1.3-2 that $P \land Q \models P \lor Q$. Thus, by the Implication Theorem, the sentence $(P \land Q) \rightarrow (P \lor Q)$ must be a tautology.
A truth table would show the same thing.

EXAMPLE 1.4-3
Show that $[(P \lor Q) \land (P \lor \neg Q)] \rightarrow P$ is a tautology; thus $(P \lor Q) \land (P \lor \neg Q) \models P$.

Solution
The following extended truth table shows that logical implication holds; in fact, the two formulas are logically equivalent.

<table>
<thead>
<tr>
<th>$(P \lor Q) \land (P \lor \neg Q)$</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>T T T T T T F T</td>
<td>T</td>
</tr>
<tr>
<td>T T F T T T T F</td>
<td>T</td>
</tr>
<tr>
<td>F T T F F F F T</td>
<td>F</td>
</tr>
<tr>
<td>F F F F F T T F</td>
<td>F</td>
</tr>
</tbody>
</table>

Using this table it is also easy to see that the associated conditional must be a tautology.

Reading Conditional Sentences

It is difficult to over-estimate the importance of the “if-then” connective, both for mathematics and ordinary discourse. Most mathematical theorems involve a conditional connective somewhere; often, it is the main connective. It is probably fair to say that mathematical sentences are typically conditional sentences, though it is a mistake to construe all mathematical sentences as being of this form, as some do. (Equational identities are not, for example.)

Conditional sentences are sometimes disguised by the use of alternative formulations. The sentence ‘if $P$ then $Q$’ can also be stated by ‘$P$ only if $Q$’ (more on this in a moment); ‘$Q$, provided that $P$’; and in other ways. There are also more hidden ways in which conditional sentences appear in mathematical theories. A very important way is in universal statements, such as ‘all differentiable functions are continuous’, which is of the form ‘all $X$s are $Y$s’. This can be reformulated in conditional format as ‘if a function is differentiable, then it is continuous’. We’ll see later in our study of PL that this form also involves a universal quantifier (‘for every thing of type $U$’), but the inner structure here remains a conditional sentence (‘if that thing is an $X$, then it is a $Y$’).

Mathematicians often formulate their results in meta-theoretical ways. They sometimes say ‘$P$ is a sufficient condition for $Q$’ or ‘in order that $Q$ it is sufficient that $P$’; and ‘$Q$ is a necessary condition for $P$’ or ‘in order that $P$ it is necessary that $Q$’. What is meant by such statements is that “$Q$ follows from $P$ (and the axioms of the theory under consideration)”. Given the Relativized Implication Theorem, it makes sense to (re)interpret such propositions as conditional sentences in the language of the theory, as $P \rightarrow Q$.

Reading the abstract sentence $P \rightarrow Q$ poses a minor language problem. Naturally, the preferred reading is ‘if $P$ then $Q$’, but the split phrasing for the connective is a bit awkward, given the order in which the symbols occur. It would be better if there was an appropriate connective word $xyz$ that could be substituted for the connective $\rightarrow$ so that $P \rightarrow Q$ could be read as ‘$P xyz Q$’. There is of course the graphic reading ‘$P$ arrow $Q$’, but this sounds artificial. ‘$P$ then $Q$’ is probably better; just drop the ‘if’ of the connective phrase.

Or, we could read $P \rightarrow Q$ as ‘$P$ only if $Q$’. This sentence is certainly equivalent to ‘if $P$ then $Q$’ in contexts where ‘if-then’ indicates the relation of logical implication. Here $Q$ is a
necessary condition for \( P \); you can’t have \( P \) without having \( Q \). Hence: ‘\( P \) only if \( Q \)’. On the other hand, ‘\( P \) only if \( Q \)’ means \( Q \) follows from \( P \). Thus: ‘if \( P \) then \( Q \)’. The only trouble with reading \( P \rightarrow Q \) this way is that many people confuse “only if” with “if”. Since these two indicate logically opposite directions, some care must be taken in using ‘only if’.

Unfortunately, using ‘implies’ as a translation for \( \rightarrow \) works all too well, nicely reinforcing the misconception we just spent some time and effort exposing. Conditional sentences are often read as implications by mathematicians. There is some measure of justification for doing this in the context of a given axiomatic theory, as we saw, but strictly speaking it is wrong in general. Regrettably, reading \( P \rightarrow Q \) as ‘\( P \) implies \( Q \)’ seems to be deeply ingrained in the mathematical psyche and may be impossible to eradicate.

**Compound Sentences and the Priority Level of Conditionality**

In writing and reading compound sentences involving \( \rightarrow \), parentheses will often be required, but they can sometimes be omitted using the priority convention we will now state. We will rank \( \rightarrow \) lower in priority than the connectives \( \neg \), \( \wedge \), and \( \vee \), as the following table indicates.

<table>
<thead>
<tr>
<th>Priority</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( \neg )</td>
</tr>
<tr>
<td>2</td>
<td>( \wedge )</td>
</tr>
<tr>
<td>3</td>
<td>( \vee )</td>
</tr>
<tr>
<td>4</td>
<td>( \rightarrow )</td>
</tr>
</tbody>
</table>

Under this convention, the compound sentences in the three examples above could have been written respectively as (1) \( (P \rightarrow Q) \vee (Q \rightarrow P) \) (as given; no savings on parentheses here since \( \rightarrow \) is of lowest priority); (2) \( P \wedge Q \rightarrow P \vee Q \); and (3) \( (P \vee Q) \wedge (P \wedge \neg Q) \rightarrow P \). To avoid potential misreadings of conditional sentences, however, it might be wise at times to use more parentheses than are strictly necessary.

We will illustrate the use of parentheses in an important example involving compound conditionals.

\[ \exists \text{ EXAMPLE 1.4-4} \]

Show that \( P \rightarrow (Q \rightarrow R) \equiv (P \wedge Q) \rightarrow R \).

**Solution**

The following 8 line truth table shows the logical equivalence.

| \( P \rightarrow (Q \rightarrow R) \) | \( (P \wedge Q) \rightarrow R \) |
| T T T T T T F T |
| T F T F F F T T |
| T T F T T F T F |
| F T T T T F T F |
| F T T F T F T F |
| F T F T T F T F |
| F T F T F T F F |
| F T F T F F F F |

This example shows that piling up the conditions \( P \) and \( Q \) in the complex conditional \( P \rightarrow (Q \rightarrow R) \) amounts to conjoining them. (Note that we had to use parentheses here to indicate which \( \rightarrow \) was to be performed first; \( \rightarrow \) is not associative – see Exercise 60). Complex nested conditionals can occur in mathematics, but mathematicians tend to formulate them in the equivalent version gotten by conjoining all the conditions, since that seems to be simpler to comprehend.

1.4-8
Conditional Sentences and Related Statements

When we gave the truth table for \( P \rightarrow Q \), we noted that a conditional sentence form could be considered logically equivalent to two other sentence forms:

\[
P \rightarrow Q \equiv \neg(P \land \neg Q) \quad \text{and also} \quad P \rightarrow Q \equiv \neg P \lor Q.
\]

The latter equivalence is simpler and maybe easier to remember, but the first equivalence is particularly useful for simplifying negated conditionals. For since \( P \rightarrow Q \equiv \neg(P \land \neg Q) \), their negations \( \neg(P \rightarrow Q) \) and \( \neg(P \land \neg Q) \) must also be equivalent. Via Double Negation, we obtain

\[
\neg(P \rightarrow Q) \equiv P \land \neg Q.
\]

This equivalence will have a role to play later on in certain proofs by contradiction.

A third sentence that is equivalent to \( P \rightarrow Q \) is itself a conditional sentence. It is formed by interchanging the two conditions and negating them. The resulting sentence \( \neg Q \rightarrow \neg P \) is called the contrapositive of \( P \rightarrow Q \):

\[
P \rightarrow Q \equiv \neg Q \rightarrow \neg P.
\]

You can easily show this equivalence by an extended truth table (see Exercise 74). A sentence and its contrapositive assert exactly the same thing, one doing it in a positive way and the other in a negative way.

What about other conditional sentences involving \( P, Q \), and their negations? Can anything be said about their logical interconnections? The converse of \( P \rightarrow Q \) is \( Q \rightarrow P \); it is gotten simply by interchanging (converting) the two sentences \( P \) and \( Q \). A sentence and its converse assert very different things; neither one implies the other. It often happens in geometry and elsewhere that a conditional statement and its converse are both true, but this is not the case in general and should never be assumed (see Exercise 75). For example, the converse of “if \( ABCD \) is a rectangle, then the diagonals \( AC \) and \( BD \) bisect one another” is false. (Why?) Beginning mathematics students sometimes confuse a statement with its converse, perhaps because they misunderstand the meaning of “if-then”. But these two conditional forms must be kept distinct or falsehood may result.

Mathematical theorems are usually far more complex than the simple conditional form \( P \rightarrow Q \) indicates. Often the antecedent (or consequent) is itself a compound sentence. For instance, the conditional may be of the form \( P \land Q \rightarrow R \). The converse of such a sentence is \( R \rightarrow P \land Q \). The two conditionals \( R \land P \rightarrow Q \) and \( R \land Q \rightarrow P \) can be considered partial converses. It often turns out in mathematics that though the full converse of a sentence cannot be proved, a partial converse can be. Fortunately, that’s often what is wanted, for many times one of the conditions (\( P \) or \( Q \)) is intended as a given that should be assumed no matter which conclusion is drawn. However, these partial converses are no more equivalent to the original sentence than the full converse (see Exercise 76). Some genuine work is required to prove them if they are true.

In a similar way you can form partial contrapositives. The partial contrapositives of \( P \land Q \rightarrow R \) would be \( P \land \neg R \rightarrow \neg Q \) and \( Q \land \neg R \rightarrow \neg P \). The interesting thing here is that both of these contrapositives as well as the full contrapositive are logically equivalent to the original conditional. Verifying this will be left as an exercise (see Exercises 77-78).

Syntax and Semantics of Biconditional Sentences

Sentences of the form ‘\( P \) iff \( Q \)’, which we have been using in the meta-language for some time now in our discussion of SL, are known as biconditional sentences. Remarks similar to the ones we made at the outset of this section to motivate the truth-functional definition of the sentential connective “if-then” could be made here to introduce the “iff” connective. Were
we to do this, we would argue that the sentential connective “iff” is intended as a sentential approximation to logical equivalence, in the sense that $P \iff Q$ iff the compound sentence ‘$P$ iff $Q$’ is a logical truth (the Equivalence Theorem below). Our treatment of biconditionals can be brief, however, since we have already discussed conditional sentences in fair detail.

The biconditional sentence ‘$P$ iff $Q$’ is symbolized by $P \leftrightarrow Q$. As with the conditional sentence, there are alternative notations available for representing the biconditional connective. Logic texts sometimes use $\equiv$; mathematicians often use the $\iff$ symbol. Since both of these are also used at times to indicate logical equivalence, which is a logical relation and not a logical operator, we have chosen the single-shafted double arrow $\leftrightarrow$ instead; it matches our use of $\rightarrow$. On the other hand, since some texts use $\equiv$ for their sentential connective “iff”, we decided earlier not to use $\equiv$ for logical equivalence, either, adopting the suggestive non-standard symbol $|=\!|$ instead.

**DEFINITION 1.4 - 2: Truth-Value Assignment for Biconditionals**

A biconditional sentence $P \leftrightarrow Q$ is true iff $P$ and $Q$ have exactly the same truth values.

The conventional truth table of $P \leftrightarrow Q$ is therefore given by the following:

<table>
<thead>
<tr>
<th>$P$</th>
<th>$Q$</th>
<th>$P \leftrightarrow Q$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$T$</td>
<td>$F$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$F$</td>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

The remarks we made about the strangeness of the truth-value assignment for $P \rightarrow Q$ apply here as well. Briefly, we will have to consider some concrete biconditionals to be true even though there is no logical relation whatsoever connecting the two conditions. As before, this is because we have designed a truth-functional connective that will yield a true sentence not only when the two sentences are logically equivalent, but also when they are logically unrelated sentences having identical truth values; otherwise $\leftrightarrow$ would not be a genuine binary logical operator mirroring logical equivalence.

On the basis of the truth-value assignment for $P \leftrightarrow Q$, the following Equivalence Theorem and Relativized Equivalence Theorem can be proved, much like the two Implication Theorems were proved above.

**THEOREM 1.4 - 3: Equivalence Theorem**

$P \models Q$ iff $|=\!| P \leftrightarrow Q$.

**Proof:**

See Exercise 92.

**THEOREM 1.4 - 4: Relativized Equivalence Theorem**

$P \models_A Q$ iff $|=\!|_A P \leftrightarrow Q$.

**Proof:**

See Exercise 93.

According to these two Equivalence Theorems, then, the sentential connective $\leftrightarrow$ is a sentential approximation to logical equivalence, though the two notions are definitely not identical. The remarks made concerning interpreting $\rightarrow$ as $|=\!|$ apply to interpreting $\leftrightarrow$ as $|=\!|$, too: such a reinterpretation gives a valid conclusion for biconditionals that are true in a given theory, but it is both misleading and wrong in general.

Biconditional sentences occur frequently in mathematics. They are most often used to state definitions (though mathematicians sometimes get lazy and use ‘if’ when they really mean ‘iff’), but they occur any time a concept is being characterized. Theorems given in ‘iff’ form are often
considered as providing alternative definitions for a concept. For example, you might define the concept of an integer being odd by saying ‘an integer is odd iff it is not even’ and then go on to prove the theorem that ‘an integer is odd iff it leaves remainder 1 when divided by 2’. This last proposition is also a biconditional and can itself be taken as a definition of being odd, in which case the original definition would become a theorem needing proof. Such situations occur repeatedly in mathematics. Where they do, mathematicians are at some liberty to choose the definition that seems most naturally suited to capture the fundamental concept for a given context or that seems to be the best deductive basis for what is to follow.

As with conditional sentences, more than one mathematical style is in use for expressing biconditional sentences. Biconditional sentences are most frequently formulated by means of ‘iff’, but at other times you will read, ‘in order that $P$, it is necessary and sufficient that $Q$’; or, more compactly, ‘such and such is a necessary and sufficient condition for so and so’. In such cases a biconditional is being asserted, and you should be able to rephrase it in standard form if necessary. Some mathematicians phrase their biconditional sentences by saying something like ‘$P$ is equivalent to $Q$’, or, ‘the following are equivalent: $P$, $Q$’. These are really statements in the meta-language, but they can be taken (via the obvious transformation) to be object language biconditionals that need to be or have been proved. This practice is legitimized by the Relativized Equivalence Theorem.

In compound sentences where one or more $\leftrightarrow$ occur along with other connectives, parentheses will generally be needed to render them unambiguous. Regarding precedence, we will take $\leftrightarrow$ as a connective of lowest priority, but on the same level as $\rightarrow$. Hence, if both $\rightarrow$ and $\leftrightarrow$ occur in a single sentence, parentheses may be required to set priority. Our final list of priorities for the main logical connectives, therefore, is the following.

<table>
<thead>
<tr>
<th>Priority</th>
<th>Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(      )</td>
</tr>
<tr>
<td>1</td>
<td>$\neg$</td>
</tr>
<tr>
<td>2</td>
<td>$\land$</td>
</tr>
<tr>
<td>3</td>
<td>$\lor$</td>
</tr>
<tr>
<td>4</td>
<td>$\rightarrow$, $\leftrightarrow$</td>
</tr>
</tbody>
</table>

**Equivalents of Biconditional Sentences**

The double arrow in $P \leftrightarrow Q$ suggests that the biconditional is somehow composed of the two sentences $P \rightarrow Q$ and $P \leftarrow Q$ (which is the same as $Q \rightarrow P$). The same thing and more is indicated by the phrase ‘if and only if’. ‘$P$ only if $Q$’ is symbolized by $P \rightarrow Q$, while ‘$P$ if $Q$’ is symbolized by $Q \rightarrow P$; the ‘and’ between ‘if’ and ‘only if’ suggests that the two sentences should be conjoined. We could thus define $P \leftrightarrow Q$ in terms of the two connectives $\rightarrow$ and $\land$ by the sentence $(P \rightarrow Q) \land (Q \rightarrow P)$; we have instead taken $\leftrightarrow$ as a primitive connective in its own right. Constructing truth tables for the sentences involved, the following equivalence is easily shown (see Exercise 79):

$$P \leftrightarrow Q \equiv (P \rightarrow Q) \land (Q \rightarrow P).$$

$(P \rightarrow Q) \land (Q \rightarrow P)$ is the principal equivalent of $P \leftrightarrow Q$, but there are three others that are related to it and are useful at times:

\[P \leftrightarrow Q \equiv -Q \leftrightarrow -P,\]
\[P \leftrightarrow Q \equiv (P \rightarrow Q) \land (-P \rightarrow -Q),\]
\[P \leftrightarrow Q \equiv (P \land Q) \lor (-P \land -Q).\]

These, too, can be shown by means of truth tables (see Exercises 80-82). Alternatively, they can be shown equivalent by stringing equivalences together. The first two equivalents follow
from the principal equivalent just stated and the equivalence between a conditional sentence and its contrapositive; the third one is shown by applying the principal equivalence, the main equivalence given above for a conditional sentence, the distributive laws for $\land$ and $\lor$, and certain absorption laws.

**EXERCISE SET 1.4**

**Problems 1-8: Sentence Symbolization**

Formulate the following sentences as conditional or biconditional sentences involving the connectives ‘$\rightarrow$’ and ‘$\leftrightarrow$’. Use whatever mathematical and logical symbolism is familiar to you.

1. $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$ if $|r| < 1$.
2. $x$ is a real number only if it is a rational number or an irrational number.
3. $|a| = a$, provided $a \geq 0$.
4. $a - b \neq b - a$ unless $a = b$.
5. In order for a function to be continuous it is sufficient for it to be differentiable.
6. A necessary condition for two lines to be parallel is that they be everywhere equidistant.
7. The condition $a_n \rightarrow 0$ is necessary but not sufficient for the series $\sum a_n$ to converge.
8. A number is prime just in case it has no proper divisors.

**Problems 9-16: Formulating Definitions**

Write down good definitions of the following terms, using ‘$\leftrightarrow$’ and any other logical and mathematical symbolism that is appropriate. Use a mathematics textbook or a dictionary to locate your definitions if you need to.

9. $\triangle ABC$ is an isosceles triangle.
10. $l$ and $m$ are perpendicular lines.
11. $n$ is a composite number.
12. $a$ divides $b$ (symbolized by $a \mid b$).
13. $c$ is a zero (root) of a function $f$.
14. $f$ is a one-to-one function.
15. $f$ is continuous at a point $a$.
16. $S$ is a subset of $T$ (symbolized by $S \subseteq T$).

**Problems 17-18: Truth Values of Statements**

Determine the truth value for the following statements if $P$ and $R$ are true and $Q$ and $S$ are false.

17. $\neg P \land (Q \rightarrow S) \lor R$
18. $P \leftrightarrow (\neg Q \lor R) \land (R \rightarrow S)$

19. Is the sentence that was used to motivate the truth-value assignment for conditional sentences – *If the sides of $A'B'C'D'$ are congruent to the corresponding sides of $ABCD$, then $A'B'C'D'$ is congruent to $ABCD$* – true or false? Explain.

**Problems 20-22: True or False**

Are the following statements true or false? Explain your answer.

20. The symbol ‘$\rightarrow$’ denotes a truth-functional connective.
21. \( P \rightarrow Q \models P \lor \neg Q \)

*22. \( P \rightarrow Q \) is logically false iff \( P \not\models Q \).

**Problems 23-25: Explanations**

Explain the following, using your own words.

23. The truth-functional assignment for \( P \rightarrow Q \)

*24. The relation between \( \rightarrow \) and \( \models \)

25. The Implication Theorem

**Problems 26-28: Problematic Formulations**

What is wrong with the following statements, which are sometimes found in texts on logic and proof? What should be said instead?

26. ‘\( P \rightarrow Q \)’ means if ‘\( P \)’ is true, then ‘\( Q \)’ is true.

*27. ‘\( P \rightarrow Q \)’ means either ‘\( P \)’ is false, or else ‘\( Q \)’ is true.

28. ‘\( P \rightarrow Q \)’ means ‘\( P \)’ implies ‘\( Q \)’.

**Problems 29-44: Tautologies and Implication**

For the following problems,

(a) determine which of the following sentences are tautologies by working out their truth values; and

(b) then state what your answer means, according to the Implication Theorem, regarding logical implication.

29. \( P \rightarrow P \lor Q \)

*30. \( P \lor Q \rightarrow P \)

31. \( P \rightarrow P \land Q \)

32. \( P \land Q \rightarrow P \)

*33. \( P \land (P \rightarrow Q) \rightarrow Q \)

34. \( (\neg P \rightarrow Q) \rightarrow (P \lor Q) \)

*35. \( (P \lor Q) \land \neg Q \rightarrow P \)

36. \( (\neg P \rightarrow P) \rightarrow P \)

37. \( P \rightarrow (\neg P \rightarrow Q) \)

*38. \( [(P \rightarrow Q) \rightarrow P] \rightarrow P \)

39. \( [P \rightarrow (Q \rightarrow R)] \rightarrow [(P \rightarrow Q) \rightarrow (P \rightarrow R)] \)

40. \( [P \rightarrow Q] \rightarrow [(P \rightarrow (Q \rightarrow R)) \rightarrow (P \rightarrow R)] \)

41. \( (P \rightarrow Q) \rightarrow [(Q \rightarrow R) \rightarrow (P \rightarrow R)] \)

*42. \( (P \rightarrow Q) \rightarrow [(P \rightarrow \neg Q) \rightarrow \neg P] \)

43. \( (P \rightarrow Q) \lor (P \rightarrow \neg Q) \)

44. \( \neg(P \leftrightarrow \neg P) \)

**Problems 45-54: Implication and Tautologies**

For the following problems,

(a) determine whether the claim of logical implication is true or false; and

(b) then state what your answer means, according to the Implication Theorem, about the associated conditional statement. [If there are multiple premises, you should conjoin them to formulate a compound conditional. See Exercise 91 below.]

*45. \( P \rightarrow Q, \neg P \not\models \neg Q \)

46. \( P \rightarrow Q, \neg Q \not\models \neg P \)
47. $P \rightarrow Q \equiv P \vee Q$

48. $P \land Q \equiv P \rightarrow Q$

49. $P \rightarrow Q \equiv P \vee Q$

50. $P \lor Q \rightarrow R \equiv P \lor R \rightarrow Q \lor R$

51. $(P \rightarrow Q) \rightarrow Q \equiv P \rightarrow Q$

52. $P \lor Q, P \rightarrow R, Q \rightarrow R \equiv R$

53. $(P \rightarrow Q) \rightarrow Q \equiv P \rightarrow Q$

54. $P \rightarrow Q \equiv P \lor Q$

55. $(P \rightarrow Q) \rightarrow Q \equiv P \lor Q$

56. $P \iff \neg Q \equiv P \lor Q$

57. $Q \rightarrow R \equiv P \lor Q \rightarrow P \lor R$

58. $\neg (P \land Q) \equiv P \rightarrow \neg Q$

59. $P \iff Q \equiv \neg P \iff \neg Q$

60. $P \rightarrow (Q \rightarrow R) \equiv (P \rightarrow Q) \rightarrow R$

61. $P \land Q \rightarrow R \land Q \equiv P \rightarrow (Q \rightarrow R)$

62. $P \lor Q \rightarrow R \land Q \equiv (P \lor Q) \rightarrow (P \rightarrow R)$

Problems 54-69: Equivalence and Tautologies

For the following problems,

(a) determine whether the claim of logical equivalence is true or false; and

(b) then state what your answer means, according to the Equivalence Theorem, about the associated biconditional statement.

54. $P \rightarrow Q \iff P \lor Q$

55. $(P \rightarrow Q) \rightarrow Q \iff P \lor Q$

56. $P \rightarrow \neg Q \iff P \lor Q$

57. $Q \rightarrow R \iff P \lor Q \rightarrow P \lor R$

58. $\neg (P \land Q) \iff P \rightarrow \neg Q$

59. $P \iff Q \iff \neg P \iff \neg Q$

60. $P \rightarrow (Q \rightarrow R) \iff (P \rightarrow Q) \rightarrow R$

61. $P \land Q \rightarrow R \land Q \iff P \rightarrow (Q \rightarrow R)$

Problems 70-71: True Conditionals and Implication

Show, as claimed in the text, that the following implications do not hold, even though the associated conditional sentences can be considered true according to the truth-functional definition of $P \rightarrow Q$. (Hint: follow the method of reinterpretation discussed in Section 1.1.) Does this fact contradict the Implication Theorem? Why or why not?

70. $0 = 1 \not\equiv \sqrt{2}$ is irrational.

71. “All triangles are equilateral” does not imply that “all rectangles are squares”.

Problems 72-82: Equivalences

Show that the following are equivalent.

72. $P \rightarrow Q \iff \neg P \lor Q$ (Show this with a truth table.)

73. $P \rightarrow Q \iff \neg (P \land \neg Q)$ (Show this with a truth table or by using De Morgan’s Laws and Double Negation.)

74. $P \rightarrow Q \iff \neg Q \rightarrow \neg P$ (Show this with a truth table.)
75. Show both by means of an extended truth table and by giving a concrete mathematical example that a conditional sentence \( P \rightarrow Q \) and its converse \( Q \rightarrow P \) are not logically equivalent; neither implies the other.

76. Show that a compound conditional \( P \land Q \rightarrow R \) is not logically equivalent to either of its partial converses, \( R \land P \rightarrow Q \) or \( R \land Q \rightarrow P \).

*77. \( P \land Q \rightarrow R \) | \( P \land \neg R \rightarrow \neg Q \)

78. \( P \land Q \rightarrow R \) | \( P \land \neg R \rightarrow \neg Q \)

79. \( P \rightarrow Q \) | \( P \land (Q \rightarrow P) \)

80. \( P \rightarrow Q \) | \( \neg P \leftrightarrow \neg Q \)

81. \( P \leftrightarrow Q \) | \( (P \land Q) \lor (\neg P \land \neg Q) \)

82. The following proposition formulates the Basic Comparison Test for infinite series of non-negative real numbers:
   Suppose that \( 0 \leq a_n \leq b_n \) for all natural numbers \( n \). Then if the series \( \sum b_n \) converges, the series \( \sum a_n \) converges, too.
   a. Formulate this sentence as a single compound conditional in two ways, and tell why they are equivalent.
   b. Beginning with the form from part a that has a conjoined antecedent, formulate the partial contrapositive that retains the same ‘given’ as the original statement. If the first statement has been shown to be true, what can you say about the second one?

84. Liouville’s Theorem in complex analysis states that a bounded, everywhere differentiable function is constant. (This is not true for real analysis.)
   a. Write this proposition using the logical apparatus of SL. Use the letters \( P, Q, \) and \( R \) for the three atomic sentences involved, and make all sentential connectives explicit.
   b. Give two equivalent formulations of the sentence you obtained in part a by writing its partial contrapositives (see Exercises 77-78).
   c. Put your SL reformulations of part b back into good mathematical English.

Problems 85 - 89: Conditional and Biconditional Equivalents for Other Connectives

Show the following are equivalent. See Exercises 1.3-31 and 1.3-32 for a description of the connectives involved.

85. \( P \lor Q \) | \( \neg (P \rightarrow Q) \)

86. \( P \rightarrow Q \) | \( [(P \lor P) \lor Q] \lor [(P \lor P) \lor Q] \)

87. Using Exercise 1.3-32 and the last exercise, determine a logical equivalent for \( P \rightarrow Q \) using only \( \nabla \).

88. Using Exercise 1.3-31, determine a logical equivalent for \( P \rightarrow Q \) using only \( \Delta \).

89. Using Exercise 1.3-31 and the last exercise, determine a logical equivalent for \( P \leftrightarrow Q \) using only \( \Delta \).

Problems 90 - 93: Proofs of Theorems

Prove the following theorems.

90. Relativized Implication Theorem: \( \mathcal{A}, P \vdash Q \iff \mathcal{A} \vdash P \rightarrow Q \).

91. Relativized Implication Theorem, Conjunctive Form: \( P_1, P_2, \ldots, P_n \vdash Q \iff \mathcal{A} \vdash P_1 \land P_2 \land \cdots \land P_n \rightarrow Q \).

92. Equivalence Theorem: \( P \vdash Q \iff \mathcal{A} \vdash P \rightarrow Q \).

93. Relativized Equivalence Theorem: \( P \vdash \mathcal{A} \vdash Q \iff \mathcal{A} \vdash P \leftrightarrow Q \).
HINTS TO STARRED EXERCISES 1.4

2. “Only if” is not the same as “if”. See page 1.4-7.
3. “Provided” indicates a conditional.
4. There are different takes on the meaning of “unless”; all involve negation in some way.
5. “Just in case” stipulates the exact situation in which the given statement holds.
6. Recall that isosceles triangles are those with two congruent sides.
7. A composite number is one that has factors other than 1 and itself.
8. \( a \text{ divides } b \) is equivalent to saying \( b \text{ is exactly divisible by } a \).
17. Remember the priority conventions given on 1.4-8 as you determine the sentence’s truth-value.
22. [No hint.]
24. [No hint.]
27. Note that “\( P \rightarrow Q \)” can be stated even if it isn’t true.
30. Review the definition of tautology and the statement of the Implication Theorem.
33. Review the definition of tautology and the statement of the Implication Theorem.
35. Review the definition of tautology and the statement of the Implication Theorem.
38. Review the definition of tautology and the statement of the Implication Theorem.
42. Review the definition of tautology and the statement of the Implication Theorem.
45. A set of premises logically implies the conclusion iff whenever all the premises are true, the conclusion is also true.
48. A set of premises logically implies the conclusion iff whenever all the premises are true, the conclusion is also true.
49. A set of premises logically implies the conclusion iff whenever all the premises are true, the conclusion is also true.
75. Find two mathematical sentences \( P \) and \( Q \) such that \( P \rightarrow Q \) and \( Q \rightarrow P \) have different truth values.
77. Use an extended truth table to show this equivalence.
1.5 Introduction to Deduction; Rules for AND

We are now familiar with the syntax and semantics of SL-sentences, and we know how to determine whether a set of premises $P$ logically implies a conclusion $Q$: write out an extended truth table and check whether $Q$ is true every time the premises $P$ are, jointly. Fortunately, this is a mechanical process a computer can do, for as the number of atomic sentences increases, the procedure becomes cumbersome and time-consuming: the number of rows in the truth table grows exponentially with the number of sentence variables involved (see Exercise 1.2-44).

Concerns with size alone would make it worth having another method to show that a conclusion follows from a set of premises. Giving a deduction (a derivation) is one such method. But there is a much deeper and more important reason for learning to do derivations: this is what is needed to pursue fields like mathematics more intelligently. One of the main tasks of logic is to provide a systematic logical framework for deductive reasoning as it occurs in mathematics, computer science, and elsewhere. Until now we have ignored deductive arguments in the sense of proofs. We’ve discussed valid arguments in the context of logical implication, but that’s quite a different matter. A sequence of sentences that merely lists a number of premises followed by a consequence is certainly not a deductive argument in the ordinary sense of the term. Simply postulating Euclid’s axioms, for example, and then concluding the Pythagorean Theorem (Proposition I.47 in Euclid’s Elements) hardly counts as a proof. No logical movement or reasoning has taken place; the axioms still need to be linked together to generate a proof. You can legitimately claim that the Pythagorean Theorem follows from the axioms (an assertion about the state of affairs holding between these propositions), but not that you have shown that it follows. A proof or deduction is offered to demonstrate that a logical consequence does in fact follow from the given set of premises using accepted modes of reasoning. Proof is what is needed for justifying and explaining new results in terms of old ones; we don’t generally use truth tables for this purpose.

Setting Up a Natural Deduction System for SL

A derivation of a sentence $Q$ from a set of sentences $P$, then, is a sequence of sentences that takes the sentences in $P$ as premises and concludes $Q$ from them by strictly applying logical rules of inference. Each sentence in such a derivation must either be an assumption or a conclusion legitimately drawn from earlier lines in the argument.

To indicate that $Q$ is derivable from $P$, we will write $P \vdash Q$ (read: “$P$ proves $Q$”). Derivability is a different notion than implication, so we use a different notation for it. Logical implication is defined in terms of truth values, while derivability requires the existence of an argument that follows stipulated rules of inference. Logic incorporates and stipulates rules of inference for creating deductions; all together they form what is called a Deduction System.

What features would we like SL’s Deduction System to have? An essential prerequisite is that each inference rule be sound; that is, if we can conclude $Q$ from $P$ via a given rule, then $Q$ must logically follow from $P$. Using extended truth tables, we will be able to show that the inference rules we choose for SL’s Deduction System are sound. Thus, we will be able to conclude in the end that if $P \vdash Q$, then $P \models Q$.

It would be wonderful if the Deduction System as a whole were also complete; that is, whenever a result $Q$ logically follows from premises $P$, it can also be proved from them. We have, of course, no a priori guarantee that a logical consequence $Q$ of a set of premises $P$ will be provable from them; this all depends upon the adequacy of the Deduction System we set up, whether we have selected sufficient rules of inference to do the job. Showing that our Deduction System for SL is complete is a far deeper matter than showing its soundness, and we will not argue for it here. However, the system we set up will, in fact, be complete. Thus,
the converse of soundness will also hold: if $\mathcal{P} \vdash \mathcal{Q}$, then $\mathcal{P} \vdash \mathcal{Q}$. Putting these two together, it follows that $\mathcal{P} \vdash \mathcal{Q}$ iff $\mathcal{P} \vdash \mathcal{Q}$.

Thirdly, our rules of inference should be *natural* and *simple*; that is, they should codify the (good) ways we ordinarily draw inferences in making deductive arguments. Trying to mirror actual practice gives us a rather loose requirement, but we will be able to derive a criterion from it nevertheless, and we will fix its meaning more precisely as we proceed.

How can we guarantee simplicity for our rules? Recall how rules of inference are used, and connect that to the *Principle of Logical Form* from Section 1.1. Rules of inference permit us either to write down an assumption or to infer a conclusion on the basis of the preceding lines. As laws that govern valid argumentation, rules of inference should depend solely upon the logical form or structure (here, the connectives) of the sentences involved. So if we want our rules for SL to be simple, we should make sure that the logical forms warranting the inference are not too complicated, making it easy to see that the conclusion follows from the premises.

**EXAMPLE 1.5 - 1**

Analyze the simplicity of inference rules associated with the following argument forms:

a. $P \lor Q, \neg Q \vdash P$

b. $P \leftrightarrow (Q \lor R), R \rightarrow \neg P \vdash \neg R$

**Solution**

a. The first implication is intuitively valid: given two alternatives $P$ and $Q$, if $Q$ is not the case, then $P$ must be. We’ve looked at this inference before: see Example 1.3 - 3.

b. The second implication is quite another matter. It is far from obvious; you may even wonder if it is valid (it is). Such an inferential connection is too complicated and specialized to adopt as a primitive rule of inference.

In order to ensure that our rules are simple and natural, then, we will require of our main rules that at most two (occasionally three) sentence variables be involved and that each rule operate on a single sentential connective, either *introducing* or *eliminating* it. Such rules are known, for obvious reasons, as *Int-Elim Rules*. We will have rules for *concluding* a conjunction, a disjunction, a negation, a conditional, and a biconditional; and rules that govern what you can *conclude from* such forms.* Such rules should thus be fairly simple and intuitively sound. In applying the rules to create deductions, you will be taking small steps, logically speaking, ones that should be convincing to everyone. Nevertheless, by combining them in sequence, you will be able to construct highly complex chains of reasoning. If the various *Int-Elim* rules are applied correctly, the entire argument will be valid.

We will also permit another class of rather natural rules; these will be *Replacement Rules* based on the equivalence between two sentences. For example, $P \lor Q \vdash Q \lor P$ obviously holds, so either disjunction can be substituted for the other one anywhere in an argument, even for a subsentence. The justification for this is clear: equivalent sentences have exactly the same truth values and so cannot be distinguished from this point of view (see Section 1.3). We will introduce a fair number of these Replacement Rules as we proceed, making sure that the equivalence is apparent and that the rules add something useful to our Deduction System.

Finally, while soundness and completeness are converse concepts of a sort, there is yet another potential feature of Deduction Systems that moves in the opposite direction from completeness. In order to obtain a complete Deduction System, we need to include enough rules governing sufficiently varied sorts of situations so that all deductive eventualities are covered. In so doing, however, we may be admitting inference rules that are redundant, rules that could be dropped without weakening the deductive power of the system overall. For the

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* The rules we will adopt are modified versions of rules that go back to S. Jaśkowski, G. Gentzen, and others of the mid-1930’s and later.
sake of economy, then, we could omit any rules whose conclusions can be generated by the other rules. In the extreme, this would lead to a completely independent set of inference rules.

However, independence is only relatively desirable, for the farther we whittle down our Deduction System, the more difficult it will be to deduce conclusions using the remaining rules. So, without multiplying rules excessively, our approach will be to accept any rule of inference that is sound, fairly simple, and seems intuitively obvious, even if it adds some minor redundancy. This will give us a moderately large system of rules to keep track of, but if they are simple and natural, this shouldn’t be problematic. The excess will be worth it; more rules will make the business of proof construction easier.

As we discuss SL’s Natural Deduction System, we will illustrate the various proof techniques by means of genuine mathematical arguments. However, because most mathematical proofs use certain extra-sentential rules of inference as well (rules governing quantifiers), we will generally not be able to give complete proofs of mathematical results using only the deduction system for SL. To compensate for this, we will also include some simple arguments from ordinary, everyday discourse, but mostly we will continue to draw upon a wide variety of symbolic arguments framed wholly within the symbolic language of SL.

**Simple Rules of Inference: Premises; Previous Propositions**

The first inference rule is the Rule of Premises, Prem for short. It permits you to assume as a line in your deduction any premise that the argument is based upon. This rule is sometimes glossed over or overlooked either because it seems so obvious or because it is tacitly incorporated into the definition of what a deduction is, but it shouldn’t be. Given a certain list of premises for an argument, you may assert any one of them, citing ‘Prem’ as your reason.

To record a deduction by a proof diagram, we will do as we suggested in Section 1.1. We will put the sentences of our deduction on the left hand side and cite the appropriate rule of inference in the right hand column as the reason for the inference. Our particular schema derives from those devised by S. Jaśkowski and F. B. Fitch; the Deduction System we will develop is known as a Fitch-style natural deduction system, though it originates with Jaśkowski.

A proof based upon premises $P$, $Q$, and $R$ and concluding $S$ would look much as indicated below: the sideline indicates the progression of the proof, the numbers left of this line label the steps of the proof, and the single underline separates the premises from the conclusions drawn from them. The conclusion of the deduction appears as the final line in the diagram. The reason given for each step should be the name of a rule of inference followed by the line numbers to which the rule has been applied in drawing the conclusion.

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<tr>
<td>$i$</td>
<td>$P$</td>
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<td>Inf Rule X</td>
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<td>$S$</td>
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You will never use such a diagram for constructing proofs in your other mathematics courses, but we will adopt it here as a tool to help us pull proofs apart and spell out the logical detail embedded in an argument. As you use proof diagrams, you should become more keenly aware both of the different proof strategies that are available to you and of the overall structure of proofs.
In a formal argument we will list all of our premises first, separating them from the rest of the proof by a horizontal line, but this is entirely a matter of bookkeeping, keeping straight precisely what the argument’s assumptions are. However, this practice does not mean that we will or should use these premises right away. When we are ready to use a sentence to make an inference, we can merely refer back to it by line number, even if it was written down quite some time before. In informal, mathematical proofs, on the other hand, we usually do not state a premise until it needs to be inserted into the argument. It is a fairly common error on the part of beginning proof-makers (of both formal and informal deductions) to want to use the premises too quickly, since they are available right away without doing any work; but this is often not the correct instinct. Even in constructing formal proofs, you shouldn’t let the fact that you’re listing the premises at the outset trick you into trying to use them prematurely. We’ll return to consider good overall proof strategy in more detail at the end of this lesson.

A closely related rule of inference might be called the Rule of a Previously Proved Proposition. Rather than proving something again that has already been proved (on the same set of premises or axioms as that given for the argument in progress!), you can just cite the abbreviations ‘Prev Propn’ or ‘Thm 3.2’ or ‘Mean Value Theorem’ or ‘Zorn’s Lemma’ or whatever identifies the given sentence as already established. This is standard mathematical procedure in developing a deductive theory. Here the axioms and definitions of the theory are considered to be the underlying premises of the entire system of propositions and are therefore not listed each time anew. Only the mathematical conclusion to be established is stated; any special conditions are thus usually incorporated into the antecedent of a conditional formulation of the proposition. Theorems are then proved by means of anything previously assumed or proved. Since we are only discussing SL in general and not developing particular axiomatic theories in detail, you will not have much occasion to use this rule of inference here (unless it comes in handy for you to use an earlier worked example or problem in doing your homework), but it is nevertheless an important rule of inference. As an inference rule, it is adopted purely as a matter of economy. It is a conservative addition to our deduction system: nothing new can be proved with it that couldn’t be proved without it.

**Introduction and Elimination Rules for AND**

The remaining rules of inference in SL’s deduction system will be introduced gradually over the next few sections. In this section we will restrict our attention to conjunction rules.

If you know \( P \land Q \) to be the case for two formulas \( P \) and \( Q \), then you are permitted to assert either \( P \) or \( Q \) by itself. The rule of inference that prescribes this is the Rule of Conjunctive Simplification, abbreviated by \( \text{Simp} \). This rule is used subconsciously in everyday reasoning. In asserting two propositions jointly, we are at the very least claiming each of them separately, so we tend to weaken the conjunction to one of the conjuncts without giving it a second thought. Such an inference is obviously sound: applying \( \text{Simp} \) to a conjunction will never lead from a true sentence to a false one, for the only time a conjunction can be true is when both of its conjuncts are true.

The \( \land \) elimination rule is schematized in the following way, either of the two forms going under the name \( \text{Simp} \):*

\[
\begin{align*}
\text{Simp} & \\
\hline
P \land Q & \quad P \\
\downarrow & \\
P & \\
\end{align*}
\begin{align*}
\hline
P \land Q & \quad Q \\
\downarrow & \\
Q & \\
\end{align*}
\]

* This rule of inference, along with all the other SL rules, can be located for easy reference on the sheet *Inference Rules for Sentential Logic.*

1.5-4
A double underline is used here to indicate that if a formula like that above the line has already occurred in some proof you are constructing, you may legitimately infer the formula below the double underline as the next step in your proof. Double underlines are not drawn in deductions; only single underlines should occur there, where they are used to separate premises from conclusions. They appear here in schematizing a rule of inference to indicate how a valid inference may be made in working a proof.* Furthermore, no sideline numbers are written in exhibiting a rule of inference, since no argument is taking place here, only a rule for how arguments may be developed.

Our introduction rule for \( \land \) is the following. If two formulas \( P \) and \( Q \) of SL occur on separate lines in a deduction prior to a given line (and in either order), you may legitimately conclude \( P \land Q \) on that line, citing the Rule of Conjunction \( (Conj) \). This rule is certainly sound, for if \( P \) and \( Q \) are both true individually, so is their conjunction \( P \land Q \). The inference schema for the rule \( Conj \) is the following:

\[
\begin{array}{c|c}
P & Q \\
\hline 
P \land Q
\end{array}
\]

The following two examples illustrate how the above rules are used to construct a deduction. They are formulated using the variables \( P, Q, \) and \( R \), but they hold for any sentences in the same overall form, regardless of their internal syntactic complexity. Each of these inferences will be covered by a Replacement Rule (discussed next), but they are derived here using \( Simp \) and \( Conj \) for the purpose of illustration.

\( \star \) EXAMPLE 1.5-2

Show that \( P \land Q \vdash Q \land P \) (commutativity of conjunction).

**Solution**

The following proof diagram establishes the claim.

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( P \land Q )</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>( P )</td>
<td>Simp 1</td>
</tr>
<tr>
<td>3</td>
<td>( Q )</td>
<td>Simp 1</td>
</tr>
<tr>
<td>4</td>
<td>( Q \land P )</td>
<td>Conj 2, 3</td>
</tr>
</tbody>
</table>

This example can obviously be turned around: \( Q \land P \vdash P \land Q \). In fact, the proof just given covers that case as well, since the two conjuncts are merely switched. So we have here an interderivability result: \( P \land Q \vdash Q \land P \), where the two-directional \( \vdash \) indicates that the formula on each side of \( \vdash \) can be derived from the one on the other side.

The next example is also one-half of an interderivability result: \( (P \land Q) \land R \vdash P \land (Q \land R) \). The second direction there requires a separate proof (see Exercise 6).

\( \star \) EXAMPLE 1.5-3

Show that \( (P \land Q) \land R \vdash P \land (Q \land R) \) (associativity of conjunction).

---

* You can keep these two diagram formats straight by noting that \( \vdash \) is present in rules-of-inference diagrams, where only logical implication or soundness is at issue, while \( \dashv \) is present in a proof diagram where deduction is taking place.
Solution
The following proof diagram establishes this claim.

<table>
<thead>
<tr>
<th></th>
<th>(P ∧ Q) ∧ R</th>
<th>Prem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P ∧ Q</td>
<td>Simp 1</td>
</tr>
<tr>
<td>2</td>
<td>P</td>
<td>Simp 2</td>
</tr>
<tr>
<td>3</td>
<td>Q</td>
<td>Simp 2</td>
</tr>
<tr>
<td>4</td>
<td>R</td>
<td>Simp 1</td>
</tr>
<tr>
<td>5</td>
<td>Q ∧ R</td>
<td>Conj 4, 5</td>
</tr>
<tr>
<td>6</td>
<td>P ∧ (Q ∧ R)</td>
<td>Conj 3, 6</td>
</tr>
</tbody>
</table>

Replacement Rules Involving AND

Replacement Rules are for substituting equivalent sentences for one another. To schematize them, we will place the symbol :: between the two equivalent sentences. Our Replacement Rules will thus take the form P :: Q,* which can be read as “P is inter-replaceable with Q”. Thus, whenever a formula F(· · · P · · · ) contains an occurrence of the sentence P, you may infer any associated formula F(· · · Q · · · ) in which Q has been substituted one or more times for P; and conversely.

We will adopt three Replacement Rules for ∧. The first two have just been mentioned: Commutation (Comm) and Association (Assoc). These rules are rarely used explicitly in informal proofs. Given the ordinary meaning of ‘and’, we can rearrange conjuncts in these ways (and in more complex ways) without affecting the truth of the sentence. So we blissfully assume it in any argument that requires it, without actually stating the inference. In formal proofs, however, we will use and cite these rules whenever they are required. These two rules are schematized in the following way:

\[
\text{Comm} (\land) \quad \begin{align*} 
P \land Q & : : Q \land P \\
\text{Assoc} (\land) & \quad P \land (Q \land R) : : (P \land Q) \land R
\end{align*}
\]

Similarly, the Idempotence Replacement Rule (Idem) for conjunction is rarely used, except in formal settings when you need to expand or contract a sentence in a certain way to set it up for making a conclusion based upon that new form. This rule is schematized as follows:

\[
\text{Idem} (\land) \quad P \land P : : P
\]

Global Proof Strategy: the Backward-Forward Method

As you might guess, there isn’t much of real interest or difficulty that can be proved from the few simple rules of inference we have formulated up to this point (see Exercises 5 and 6); our Deduction System is certainly not complete. But rather than move on to rules for another connective, we will pause to reflect on overall proof strategy. Because this is important, you should read it carefully now, but you will want to come back to reread it once you have progressed further into the Deduction System for SL. We will illustrate our ideas with the examples worked above.

The first prerequisite for constructing a good argument is an obvious one and sounds trite, but it bears stating: keep in mind what the premises P are and what the conclusion Q is.

* This notation is due to Bergmann, Moor, and Nelson in The Logic Book (1980). The symbol :: is used in mathematical circles to indicate sameness of ratios.
Beginning proof-makers occasionally get confused about what it is they are trying to prove and what they are allowed to assume in the process. You may finish a proof to your satisfaction, only to discover later on that you proved something different from what was asked. Or you may get disoriented and end up constructing a circular argument, in which you assume along the way something very close to the conclusion you are supposed to end up with. Nobody likes to think they are stupid enough to do this, yet everyone does it sometime or other. Especially when you are constructing long, complicated, informal mathematical arguments or when you are in the midst of axiomatically developing the formal theory of a field you are already quite familiar with, it is easy to get mixed up like this. Before you know it, you’ve constructed a ‘proof’ that uses as given what you’ve never deduced and so have no right to use. The best way to avoid this sort of argument in an informal setting is to continually remind yourself, using verbal indicators such as ‘have’ and ‘want’, what it is you already know and what it is you still need to show.

Assuming you have a firm hold on the premises and conclusion, there are two or three basic approaches you can follow in constructing a proof. You can either work forward from the premises to their consequences; or you can work backward from the conclusion to sentences from which it follows; or you can do a mixture of both. If you are working forward from the premises, you should continually ask yourself the question “so what?” (which does not mean “who cares?” here), while if you are working backward from the conclusion, you should keep asking yourself “why is that?” or “how could that be?” Your ultimate goal is to connect the two strands together by a logical chain of reasoning that proceeds from the premises to the conclusion; discovering (and communicating) how they link up can be done either in a forward or a backward or a part-forward, part-backward mode.

Think of a proof as a journey from a set of premises to a conclusion. To take such a trip, you must map out a route, a path of sentences that lead from the premises to the conclusion. Both the starting point (the premises) and the destination (the conclusion) are important to keep in mind in mapping out this path; you can’t ignore either one and do the job. You can start plotting your course at the beginning and move forward, but then you must keep in view your destination so that you end up where you want to be. Or you can start making your route from the end and go backward, but then you need to know where the starting point is so that you can link up to the beginning. Alternatively, you can begin at both ends and meet somewhere in the middle.

We will call this combined overall proof strategy the Backward-Forward Method of Proof Analysis.* As a general rule of thumb in constructing a proof, it is probably best to begin by backtracking from the conclusion: ask yourself what kind of sentence you wish to prove and how that can be gotten from the other sentences you’re given. The advantages of this approach include the following.

In the first place, this tactic is goal-focused; it keeps your attention focused on what needs to be done, and it helps you in the end to see how the various premises enter into the proof at different stages. If you begin instead with the premises, it will often not be clear to you which one(s) you should work with first and which one(s) you should leave aside for the moment to combine with a later result. This is particularly true in developing an axiomatic theory, where all the axioms function as potential premises, though you may not know which ones are relevant for a particular theorem. By focusing on the conclusion, you will avoid drawing conclusions that are true but irrelevant to your purpose. In proceeding backwards, you will of course have to keep the initial premises in mind, and you will especially have to take care to remember that ‘wanting’ something is different than ‘having’ it, but if this is done, starting out with the conclusion and looking for sufficient conditions is beneficial.

In the second place, the attitude engendered by this Backward-Forward Method of Proof

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* Daniel Solow in How to Read and Do Proofs (6th edition, 2013) uses a similar term, though he misconstrues it as a particular method for proving conditional sentences. The ideas behind this method have long been used for constructing proofs.
Analysis is helpful for pursuing mathematical research. Suspecting the truth of a certain proposition, mathematicians usually begin with certain preconceived notions of why such a result must be true, but as they continue their investigation, they often modify their original hypotheses, weakening or strengthening them as they discover more clearly just what premises the proposition will follow from. A joint Backward-Forward approach helps ferret these things out. Rarely, if ever, do mathematicians simply adopt a set of axioms and sit down to see what they might entail, using a purely Forward mode of proof. They invariably work from conjectures and try to link them up with what they know about the field they’re working in, using a Backward-Forward approach.

Finally, in proving a tautology (something a mathematician never does outside of a logic course), the only strategy available is the Backward Method of Proof Analysis, since the conclusion, being a tautology, is true on the basis of no premises whatsoever. Having nothing to work forward from, you must analyze how such a conclusion can come about, thus engaging the Backward mode. Such an argument will generally contain temporary assumptions, as we will see, and so again the Forward direction will come into play to give a joint Backward-Forward Method, but the proof must be started by using the Backward mode.

Unfortunately, the Backward Method and the Backward-Forward Method are often neglected by beginning proof-builders. To reiterate what we said earlier, when one begins proving theorems the native instinct seems to be to use what you’ve been given or already know since that’s close at hand – free, as it were. But this is often the wrong tack to take. Until you develop good proof-technique instincts, then, you may have to consciously remind yourself to examine the conclusion first and ask, “How could I prove such a statement? What would I need to know to get it? What other results apply to such a situation?” With these questions uppermost in your mind, you can then review the premises, definitions, and earlier propositions to see which, if any, seem to hold promise for getting such a result.

In the end, regardless of whether you use a Backward, a Forward, or a mixed Backward-Forward approach in constructing a proof, your principal guide on the level of logic* should be the logical form or structure of the sentences involved. The main connective determines what sort of sentence a given proposition is, and that in turn determines what can be done with it. This is particularly important in working with the abstract sentences of SL, but it is also helpful in trying to construct a mathematical proof. To draw conclusions from your premise set, you must use whatever rules of inference pertain to those sentence forms. And to locate sentences from which the conclusion will follow, you must try to use whatever rules of inference permit you to draw a conclusion of that type. Being thoroughly acquainted with the rules of inference that relate to the various sentence forms involved will help you devise a deduction. You may instinctively draw certain conclusions from the premises you are given without knowing any systematic logic, particularly if you are familiar with the field (the meaning of the propositions), but if you run stuck and don’t know what to do, or if the field is quite abstract and unfamiliar, as is the case in certain advanced mathematics courses, an analysis of the logical form involved may be just what you need to get going.

Let’s make our remarks more concrete now, using what little we have developed so far. Working forward from a set of premises that contain a conjunction among them, for example, you will want to see whether concluding either of the two conjuncts by means of the elimination rule Simp would help move you toward the final conclusion. Working your proof backwards, on the other hand, if the conclusion you’d like to get is a conjunction, you will want to see whether you can prove each of the conjuncts separately. The introduction rule Conj will then generate the conjunction. It is important to keep both of these strategies in mind as you are generating a proof: you want to ask what can be derived from the premises given, but also how you can arrive at the conclusion.

* Naturally, more than logic is usually involved in creating mathematical proofs; intuitions about how the mathematical ideas are related and which other ones might be pertinent are also important for proof construction, as we will see at the end of the next lesson. Here we’re only focusing on what logic can contribute.
The following example illustrates in a simple way how the Backward-Forward Method of Proof Analysis can be used to get started on a proof. We will revisit Example 1.5-3.

\textbf{EXAMPLE 1.5-4}

Use the Backward-Forward Method to construct a proof of \(P \land (Q \land R)\) from \((P \land Q) \land R\).

\textbf{Solution}

\textbf{Proof Analysis}

\((P \land Q) \land R\) is our premise, and we want to conclude \(P \land (Q \land R)\).

The conclusion is a conjunction of \(P\) with \(Q \land R\). If we get these two conjuncts separately, we can conjoin them by \textit{Conj}. At this point, though, neither one follows immediately from the premise, so we continue with our backward analysis.

In order to get \(Q \land R\), it suffices to get \(Q\) and \(R\) separately.

This is as far as our backward analysis can take us: we must try to get \(P\), \(Q\), and \(R\) individually.

\textbf{Proof Construction}

We now begin with our premise, working in a forward direction.

\(R\) can be gotten from \((P \land Q) \land R\) by \textit{Simp}; this yields \(P \land Q\) as well.

We can then further simplify \(P \land Q\) to obtain both \(P\) and \(Q\).

Now we’re ready to put everything together in order.

We first conjoin \(Q\) and \(R\) to get \(Q \land R\).

Then, conjoining this with \(P\), we obtain our conclusion: \(P \land (Q \land R)\).

If you check the formal proof given in Example 3 above, you’ll see that this sequence of steps is exactly what we did there.

This example show how to tackle making a deduction in SL. Matters will get much more complex as we proceed, but the general proof-construction procedure embodied in the Backward-Forward Method of Proof Analysis remains the same and is at bottom just simple, good common sense: determine where you want to end up, assess what you need to get there, and then see what you’re given to start with.

\textbf{EXERCISE SET 1.5}

\textit{Problems 1-4: Soundness of Inference Rules}

Show that the following Int-Elim and Replacement Rules are sound by means of extended truth tables.

1. Soundness of Simplification: \(P \land Q \models P, Q\)

2. Soundness of Conjunction: \(P, Q \models P \land Q\).

3. Soundness of Commutativity: \(P \land Q \models Q \land P\)

4. Soundness of Associativity: \(P \land (Q \land R) \models (P \land Q) \land R\)

\textit{Problems 5-6: Proof Analysis}

Construct a Backward-Forward Proof Analysis and then give a formal proof to show the following.

5. \(P \land (Q \land R) \models P \land R\)

6. The Associative Law for Conjunction

a. \(P \land (Q \land R) \models (P \land Q) \land R\). Work this using only the Int-Elim Rules for conjunction.

b. Repeat part a, using any rules for conjunction.
Problems 7-9: Logical Implication and Derivability

*7. Derivability, Derivations, and Validity
   a. Tell why $P \land Q$ cannot be derived from $P \land (Q \land R)$ in one step by means of Simp.
   b. Show that $P \land (Q \land R) \vdash P \land Q$, using any inference rules from this section.
   c. By analyzing the derivation you used in part b, explain why $P \land (Q \land R) \vdash P \land Q$.

8. Logical Implication and Derivability
   If we were feeling giddy and decided that after this section we had enough rules of inference in our Deduction System for SL, which of the following eight claims would hold? Explain your answers.
   i. $P, Q \vdash P \land Q$; $P, Q \vdash P \land Q$.
   ii. $P \land Q \vdash P \lor Q$; $P \land Q \vdash P \lor Q$.
   iii. $P \lor Q \vdash P \land Q$; $P \lor Q \vdash P \land Q$.
   iv. $(P \lor Q) \land \neg Q \vdash P$; $(P \lor Q) \land \neg Q \vdash P$.

9. Axiomatic Systems and Derivations Using Previously Proved Propositions
   a. Suppose that $A \vdash P$ and $P \vdash Q$, where $A$ is some set of sentences in SL and $P$ and $Q$ are likewise sentences in SL. What do you think you should be able to conclude about the relation holding between $A$ and $Q$? Explain your answer.
   b. Comment on the soundness and conservative character of the rule of inference that permits you to use previously proved propositions in developing an axiomatic theory.

Problems 10-13: True or False

*10. Sound inference rules are only applied to true premises.

11. Replacement Rules are special types of Int-Elim Rules.

12. The set of inference rules we will choose for SL’s Deduction System will be a set of independent rules.

13. Premises are listed at the outset of a deduction, but they are often not used until later in the proof.

Problems 14-15: Explanations

Explain the following, using your own words.

14. Tell how to proceed in constructing a deduction of a conclusion from a set of premises (the Backward-Forward Method of Proof Analysis).

15. What is the difference between a sentence $Q$ being a logical consequence of a set of sentences $\mathcal{P}$ and its being deduced from $\mathcal{P}$? What different means are available in SL for showing these two things? Which relationship depends upon the particular Deduction System chosen?

Problems 16-19: Drawing Conclusions from Premises

What conclusions can be drawn by means of the rules of inference available so far from the following sets of premises? List them or describe what all such sentences look like. (Give this some thought; don’t be too hasty!)

16. $P, Q$

17. $P \land Q$

18. $P \land Q \rightarrow R$

19. $P, P \rightarrow Q$

Problems 20-23: Finding Premises for Conclusions

Find several premise sets that can be used to yield the following conclusions, assuming the rules of inference given in this lesson. Do not list any extraneous premises (premises that do not figure in the deduction) or premises containing additional sentence variables.

20. $P \land Q$
21. \((P \land Q) \land \neg R\)
22. \((P \lor Q) \land R\)
23. \(P \rightarrow Q \land R\)

Problems 24-30: Hofstadter’s MIU-System

The following MIU-System is due to Douglas R. Hofstadter in Gödel, Escher, Bach: an Eternal Golden Braid, pp. 33-43 and 259-61.*

The alphabet for this system contains only the letters M, I, and U. Words in this language, such as MUMMI, are formed by concatenation; new words can be made from old words using the following four rules (X and Y stand for any words in the language):

1. Any word ending in an I can be extended by a U: XI \(\rightarrow\) XIU.
   For example: MUMMI generates MUMMIU according to this rule.

2. A word beginning in M can be extended by duplicating what follows: MX \(\rightarrow\) MXX.
   For example: MUMMI generates MUMMIUMMI according to this rule.

3. III may be replaced by U: XIIIY \(\rightarrow\) XUY.
   For example: UMIIIMU generates UMUMU according to this rule.

4. UU may be deleted: XUUY \(\rightarrow\) XY.
   For example: MUUI generates MI according to this rule.

*24. Derive MUIIU from MIUUI. Indicate which rules you apply as you pass from one word to the next.

*25. Show that MUIIU and MUI can each be derived from the other using the above rules.

*26. Derive the word MUIIU from the word MI using the above rules.

*27. Derive MUUIU from MI using the above rules.


EC 29. **Puzzler:** derive MU from MI using the above rules, or show it cannot be done (Hofstadter’s MU puzzle).

EC 30. **Further Exploration:** If you are given MI as your initial word, what sorts of words can you generate from it by means of the above rules? Can you find any necessary conditions for these words (“all words generated by MI must be such and such”)? Any sufficient conditions (“if a word is such and such, then it is generated by MI”)? Any necessary and sufficient conditions?

Problems 31-34: Mathematical Problems Involving Conjunction

The following problems illustrate how conjunction may be involved implicitly in a mathematical statement.

31. A number \(k\) is composite iff there exist numbers \(m\) and \(n\) such that \(m \neq \pm 1 \land n \neq \pm 1 \land mn = k\).
   a. What separate results must be shown in order to show that \(k\) satisfies this definition of being composite? What rule of inference must then be used on them?
   b. If you know that a number \(k\) satisfies the above definition of being composite, what individual sentences can you conclude? Which rule of inference is used for this?

32. Rewrite the statement ‘the hypotenuse of a right triangle is longer than either leg’ to exhibit the logical connectives involved. What conclusions can be drawn from this statement? What rule(s) of inference treated in this lesson justify these conclusions?

33. Rewrite the statement ‘three distinct points \(P, Q,\) and \(R\) are non-collinear’ to exhibit the logical connectives involved. Does the ‘and’ given in the original statement indicate conjunction of sentences? What rule of inference allows you to conclude that \(P, Q,\) and \(R\) are distinct points?

*34. If you are given the two premises \(-1 < x < 1\) and \(z < x,\) what new double inequality can you conclude? Write down an argument in proof diagram format establishing your result, citing the appropriate rules of inference. Recall that \(a < b < c\) is a conjunction in disguise.

Problems 35-36: Inference Rules in Mathematical Proofs
Analyze the following proofs as directed.

35. Analyze the following set-theoretic proof, pointing out where the rules of inference governing conjunctions enter in.

**Theorem: Subset Property of Intersections**
Given two set \( S \) and \( T \), their intersection \( S \cap T \) is a subset of both \( S \) and \( T \); in symbols, \( S \cap T \subseteq S \) \( \land \) \( S \cap T \subseteq T \).

**Proof:**
1. Suppose \( x \) is any element of \( S \cap T \); in symbols, \( x \in S \cap T \).
2. Then by the definition of \( S \cap T \), we know that \( x \in S \) \( \land \) \( x \in T \).
3. Thus \( x \in S \); and so \( S \cap T \subseteq S \) by the definition of being a subset.
4. Similarly, \( x \in T \); and so \( S \cap T \subseteq T \), too.
5. But then \( S \cap T \subseteq S \) \( \land \) \( S \cap T \subseteq T \).

36. Analyze the following informal proof, pointing out where it is proceeding in the forward direction and where it is proceeding in the backward direction. Then rewrite it in standard form, in a completely forward direction.

**Theorem: Multiplicative Property of Zero**
\( 0 \cdot a = 0 \) for all integers \( a \).

**Proof:**
It suffices to show that \( 0 \cdot a = 0 \cdot a + 0 \cdot a \), since cancelation holds for the set of integers under addition.
But \( 0 \cdot a + 0 \cdot a = (0 + 0) \cdot a \)
\[ = 0 \cdot a \text{ because } 0 + 0 = 0. \]
HINTS TO STARRED EXERCISES 1.5

2. [No hint.]

5. Use the main connectives to start working with both conclusion and premise.

7. a. *Simp* separates one side of the main connective from the other.
   b. Both *Simp* and *Conj* are needed for this derivation.
   c. [No hint.]

10. [No hint.]

11. [No hint.]

17. Remember that both Int-Elim Rules and Replacement Rules were given in this section.

21. [No hint.]

24. Start by deleting *UU* (rule 4).

25. For going from *MUI* to *MUIIU*, you might start with rule 1. To go from *MUIIU* to *MUI*, rule 2 gives a good start.

26. [No hint.]

27. Use rule 2 to get started.

28. Start by applying rule 2 more than once.

34. Use the Int-Elim Rules for ∧ to arrive at a new double inequality.

35. In which lines has an “and” been added (*Conj*) or dropped (*Simp*) from an earlier line?
1.6 Elimination Rules for *IF-THEN* and *IFF*

Among all the sentential rules of inference, the most important ones for mathematics are those related to the “if-then” connective. This is because, as noted in Section 1.4, mathematical propositions are typically conditional statements of one sort or another.

In this section we will consider the simpler inference rules that govern conditional and biconditional sentences. These include the main Elimination Rules and some simple Replacement Rules; the principal Introduction Rules and some additional Replacement Rules for conditionals and biconditionals will be postponed until the next section.

**Positive Elimination Rule for *IF-THEN*: Modus Ponens**

Regardless of how a Deduction System for SL is chosen, the rule we are about to discuss is invariably included because it is extremely useful for making deductions. This rule gets used repeatedly in ordinary discourse, particularly in concluding that something is true for a given object because it is true in general for all objects of that type. An argument of the form “all Xs are Ys, Z is an X; therefore Z is a Y” asserts a conclusion regarding Z because it holds for all Xs. Formulating the first sentence of this argument in conditional form,* we obtain the argument “if Z is an X, then Z is a Y; Z is an X; therefore Z is a Y”.

Consider a simple example from number theory. If a number is even, then its square is also even. Thus, given a particular even number n, we conclude that its square $n^2$ is also even. In presenting this argument, we might merely say, “n is even; therefore $n^2$ is even”, but this is obviously not the whole story. If someone were to ask why it follows that $n^2$ is even, we would probably assert the conditional sentence we thought was known: “because, if any number is even, then so is its square.” Once this additional premise is made explicit, the form of the argument and the rule of inference being used may be analyzed (though the questioner may want to pursue further why the new premise is true if there was some question to begin with). Spelled out in detail, the argument would be: “if a number is even, then its square is even; but the number n is even; therefore $n^2$ is even.”

A typical example from geometry where this rule is implicitly used over and over again is the following. A given triangle $\triangle ABC$ can be shown to have base angles $\angle A$ and $\angle B$ congruent by showing that the opposite sides $BC$ and $AB$ are congruent. Why does this follow? Because of the proposition that states “if a triangle is isosceles, then the base angles opposite the congruent sides are congruent.” Combining these two propositions, we may conclude the desired result on the basis of the same rule of inference as before.

The form of these two arguments can be made more conspicuous by taking P to stand for ‘n is even’ or ‘$\triangle ABC$ is isosceles’ and Q to stand for ‘$n^2$ is even’ or ‘$\angle A \cong \angle B$’. We then see that Q is being inferred on the basis of the two premises $P \rightarrow Q$ and P. Thus, we have the following inference rule, traditionally called *Modus Ponens* (MP). It is obviously a sound rule of inference, given the truth functional definition of $\rightarrow$; for if $P \rightarrow Q$ and P are both true, Q must be true as well (see Exercise 1).

\[
\text{MP} \quad \quad P \rightarrow Q \\
\hline
\text{P} \\
\hline
\text{Q}
\]

The Latin term *Modus Ponens* and its mate, *Modus Tollens*, discussed next, have a long tradition behind them in logical circles going back to medieval times, so we will also use these

---

* The full logical form of this sentence involves universal quantification, which is part of Predicate Logic. We will discuss this in Chapter 2.
terms, though Positive and Negative Detachment are more descriptive, for we can view each conclusion as arising from the conditional sentence by a process of detachment.

Mathematicians occasionally assert of a given object that it satisfies the condition of a certain theorem and that therefore something else is also true about it. This is just a case of Modus Ponens. We can thus view this rule of inference as affirming the antecedent \( P \) of an already known conditional sentence \( P \rightarrow Q \) in order to draw the consequent \( Q \) as an unconditional conclusion.

The related argument form known as affirming the consequent, however, is a fallacy (see Exercise 9a). Though it seems to be commonly used in everyday arguments (especially in the world of advertisement), strictly speaking, it yields an invalid argument form. We cannot conclude \( P \) to be the case from knowing that \( P \rightarrow Q \) holds and that \( Q \) is the case. It is possible for the conclusion \( P \) to be false even if both the premises are true.

Such argumentation does have a place, though it is not conclusive demonstration. Scientific theories are built up by trying to find conditions \( P \) that explain various known phenomena \( Q \). If \( P \) seems to account for this behavior and if it predicts other instances of \( Q \) that are also found to hold, this is taken as evidence for the correctness of the explanations. Yet affirming the consequent is not deductively valid; there is always the possibility a counterinstance will be discovered or an alternative explanation will be found that predicts the facts better or explains more phenomena in a unified manner, and then the new explanation will be accepted over the other. \( P \rightarrow Q \) should not be taken to assert an exclusive causal connection between \( P \) and \( Q \).

In constructing proofs that use Modus Ponens we will proceed like we did before, citing ‘\( MP \, i, j \)’ in the right hand column of line \( k \) whenever a conclusion \( Q \) has been inferred there on the basis of \( P \rightarrow Q \) and \( P \) occurring earlier in the proof on lines \( i \) and \( j \). The order in which the conditional and its antecedent occur in the argument is, of course, irrelevant to the inference being made. The following simple example illustrates the use of \( MP \).

\[ \text{EXAMPLE 1.6-1} \]

Show that \( P \land Q, P \rightarrow R, Q \land R \rightarrow \lnot S \vdash R \land \lnot S \).

**Solution**

The following proof diagram establishes the claim.

<table>
<thead>
<tr>
<th></th>
<th>( P \land Q )</th>
<th>( P \rightarrow R )</th>
<th>( Q \land R \rightarrow \lnot S )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>( P )</td>
<td>Simp 1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>( R )</td>
<td>MP 2, 4</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>( Q )</td>
<td>Simp 1</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>( Q \land R )</td>
<td>Conj 6, 5</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>( \lnot S )</td>
<td>MP 3, 7</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>( R \land \lnot S )</td>
<td>Conj 5, 8</td>
<td></td>
</tr>
</tbody>
</table>

It should be stressed that *Modus Ponens*, like all Int-Elim Rules, is meant to be applied to whole sentences occurring earlier in the proof, not merely to parts of a sentence. Misapplying Int-Elim Rules in this way usually leads to trouble. For example, concluding \( Q \) on the basis of \( P \) and \( (P \rightarrow Q) \rightarrow R \) is not only improper but invalid: \( Q \) does not follow from these two sentences (see Exercise 11); we cannot apply *Modus Ponens* to the component sub-sentence \( P \rightarrow Q \). Replacement Rules may be applied to component parts, but not Int-Elim Rules.
Negative Elimination Rule for IF-THEN: Modus Tollens

A second Elimination Rule for the “if-then” connective is known as Modus Tollens (MT) or Negative Detachment. This rule proceeds as follows. Given a conditional sentence \( P \to Q \) and the sentence \( \neg Q \), we can conclude \( \neg P \). Alternatively, given \( P \to \neg Q \) and \( Q \), which contradicts the consequent \( \neg Q \), we can conclude \( \neg P \).

Since both forms proceed similarly, we will consider them as two forms of Modus Tollens. Schematically, we have the following.

\[
\begin{array}{c|c|c}
\text{MT} & P \to Q & \neg Q \\
\hline
\neg P
\end{array}
\]

To illustrate this rule, consider the same conditional sentence as given above, ‘if \( n \) is even, then \( n^2 \) is even’, along with the negation ‘\( n^2 \) is not even’. From these two conditions we can conclude by Modus Tollens that ‘\( n \) is not even’.

Modus Tollens proceeds by denying the consequent. If the consequent \( Q \) of a conditional sentence \( P \to Q \) is not the case, then given that the conditional sentence itself actually is the case, the antecedent \( P \) cannot be the case. Such an inference is sound, for if the premises \( P \to Q \) and \( \neg Q \) are true, the conclusion \( \neg P \) must also be true (see Exercise 2). Or, to give an argument for its validity to someone familiar with Modus Ponens: if \( P \) were the case, then according to Modus Ponens, \( Q \) would have to be the case, too, but it’s not; \( \neg Q \) is. Thus \( \neg P \) must be the case. This sort of argument uses Proof by Contradiction, a strategy we will investigate in some detail in Section 1.8.

The rule gotten by denying the antecedent, on the other hand, is a fallacy, though it, too, often appears in ordinary discourse. If we know that \( \neg P \) and \( P \to Q \) are the case, we cannot conclude that \( \neg Q \) is the case. The premises can both be true and yet the conclusion false; such an argument is therefore invalid (see Exercise 9b).

We will illustrate the use of MT in the following example, which also uses MP.

\*\* EXAMPLE 1.6-2 \*\*

Show that \( P \to (Q \to R) \), \( P \), \( \neg R \) \( \vdash \neg Q \).

\textbf{Solution}

The following proof diagram establishes the claim. Note that step 5 cites steps 3 and 4 in reverse order to indicate how these steps correspond to the sentence forms given in the rule schema for MT. Being this picky is not strictly necessary unless you’re a computer following an algorithm; either order is permissible in your written deductions.

\[
\begin{array}{c|c|c|c|c|c}
1 & P \to (Q \to R) & \text{Prem} \\
2 & P & \text{Prem} \\
3 & \neg R & \text{Prem} \\
4 & Q \to R & \text{MP 1, 2} \\
5 & \neg Q & \text{MT 4, 3} \\
\end{array}
\]

Positive Elimination Rule for IFF: Biconditional Elimination

Biconditional sentences occur, as we remarked in Section 1.4, in a number of places in mathematics, but probably most frequently in definitions. We will therefore couch our discussion
about the Elimination Rules for biconditional sentences in terms of mathematical definitions, though these rules can be applied to any biconditional sentence whatsoever.

In defining a property or relation in mathematics, we usually say that an object has a certain property or that several objects have a certain relation iff such and such is the case. For example, the property of a number being even is defined by the biconditional sentence ‘n is even iff n is divisible by 2’. The binary relation of two lines being parallel is defined by ‘l and m are parallel iff l and m have no points in common’.

Definitions tell how terms are to be used. Knowing that an object is a so and so, we can then conclude that it has such and such a property based on its definition. Or, knowing that several objects stand in a certain relation to one another, we can conclude that they have the defining property for that relation, since that is what it means to be in that relation. On the other hand, if we discover that an object or group of objects have such and such a property, we can conclude that they are so and so’s or that they have a certain relation to one another. We can thus either replace a defined term by its definition (expanding via the definition), or we can replace a definition with the term it defines (abbreviating by means of the defined term).

Drawing these sorts of conclusions from definitions occurs so automatically we are usually not even aware of making an inference. Nevertheless, such conclusions depend upon the rule of inference called Biconditional Elimination (BE), the positive detachment or Elimination Rule for biconditionals. This rule allows us to conclude Q from P ↔ Q and P. Because P and Q play the same roles in P ↔ Q, we can also conclude P from the biconditional P ↔ Q along with Q. Thus, we have the following two forms for Biconditional Elimination:

BE

\[
\begin{array}{c}
P \leftrightarrow Q \\
\hline
P \\
Q
\end{array}
\quad
\begin{array}{c}
P \leftrightarrow Q \\
\hline
Q \\
P
\end{array}
\]

Biconditional Elimination is obviously a sound rule of inference. Showing this will be left as an exercise (see Exercise 3). The next example illustrates how BE is used in a deduction.

**EXAMPLE 1.6-3**

Show that \((P \leftrightarrow Q) \land R, R \rightarrow P \vdash Q\).

**Solution**

The following proof diagram establishes this result.

<table>
<thead>
<tr>
<th></th>
<th>((P \leftrightarrow Q) \land R)</th>
<th>Prem</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>((P \leftrightarrow Q) \land R)</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>((P \leftrightarrow Q) \land R)</td>
<td>Prem</td>
</tr>
<tr>
<td>3</td>
<td></td>
<td>Simp 1</td>
</tr>
<tr>
<td>4</td>
<td>((P \leftrightarrow Q) \land R)</td>
<td>Prem</td>
</tr>
<tr>
<td>5</td>
<td>((P \leftrightarrow Q) \land R)</td>
<td>Simp 1</td>
</tr>
<tr>
<td>6</td>
<td>((P \leftrightarrow Q) \land R)</td>
<td>MP 2, 4</td>
</tr>
<tr>
<td>7</td>
<td>((P \leftrightarrow Q) \land R)</td>
<td>BE 3, 5</td>
</tr>
</tbody>
</table>

**Negative Elimination Rule for IFF: Negative Biconditional Elimination**

We had two types of detachment rules for conditional statements, a positive one and a negative one. The same holds here. Besides the positive rule of Biconditional Elimination, there is Negative Biconditional Elimination (NBE). If \(P \leftrightarrow Q\) is the case but either \(P\) or \(Q\) is not (that is, either \(\neg P\) or \(\neg Q\) is the case), then the other sentence also is not the case.
NBE is a sound rule of inference; both P and Q must have the same truth value if P ↔ Q is true (see Exercise 4). Schematically we have the following for NBE:

\[
\begin{array}{c|cc|c}
\text{NBE} & P & Q & \neg P \\
& \neg Q & & \\
\end{array}
\]

These rules are just what we need when we know that something is not a so and so or when we know that such and such is not the case. We can then negate the other part of the biconditional that characterizes so and so's or objects with such and such a property or relation and then continue our deduction with that negation as our next line. They also come in handy when we wish to prove a negation. In order to show that something is not the case, we might show that its mate in an already established biconditional (possibly a definition) also is not the case. For example, in order to prove that \( n^2 \) is not even, we can show that \( n \) is not even, for \( n^2 \) is even iff \( n \) is even. This is an application of NBE.

\[\Box \text{EXAMPLE 1.6 - 4}\]

Show that \( P \rightarrow (Q \leftrightarrow R), P \land \neg Q \vdash \neg R \).

\textit{Solution}

The following proof diagram establishes this result.

\[
\begin{array}{c|lc}
1 & P \rightarrow (Q \leftrightarrow R) & \text{Prem} \\
2 & P \land \neg Q & \text{Prem} \\
3 & P & \text{Simp 2} \\
4 & Q \leftrightarrow R & \text{MP 1, 3} \\
5 & \neg Q & \text{Simp 2} \\
6 & \neg R & \text{NBE 4, 5} \\
\end{array}
\]

\textit{Chaining Conditionals and Biconditionals}

So far we have no rules that will allow us to infer a conditional or biconditional sentence as the conclusion of an argument. The main proof techniques for doing this will be explored in Section 1.7, but here we will consider the situation when two conditionals or biconditionals are chained together to deduce such a conclusion.

The most basic rule of this sort asserts the transitivity of the \( \rightarrow \) connective: it concludes \( P \rightarrow R \) from \( P \rightarrow Q \) and \( Q \rightarrow R \). The traditional name for this argument form is \textit{Hypothetical Syllogism (HS)}). This is obviously a valid argument form (see Exercise 5). The schema for this rule is the following:

\[
\begin{array}{c|cc|c}
\text{HS} & P & Q & R \\
& & Q \rightarrow R & \\
& & \rightarrow R & \\
\end{array}
\]

\[\Box \text{EXAMPLE 1.6 - 5}\]

Show that \( P \land S, P \rightarrow \neg R, \neg R \rightarrow \neg Q \vdash S \land \neg Q \).
**Solution**

The following proof diagram establishes the claim.

1. $P \land S$  
   Prem
2. $P \rightarrow \neg R$  
   Prem
3. $\neg R \rightarrow \neg Q$  
   Prem
4. $P \rightarrow \neg Q$  
   HS 2, 3
5. $P$  
   Simp 1
6. $\neg Q$  
   MP 4, 5
7. $S$  
   Simp 1
8. $S \land \neg Q$  
   Conj 7, 6

There is a biconditional counterpart to **Hypothetical Syllogism** as well. This rule has no standard name; we will call it **Biconditional Transitivity** (BiTrans). This rule of inference is schematized as follows.

\[
\text{BiTrans} \quad \begin{array}{c}
P \leftrightarrow Q \\
Q \leftrightarrow R \\
P \leftrightarrow R
\end{array}
\]

This rule says that if $P \leftrightarrow Q$ and $Q \leftrightarrow R$ both hold, then we may conclude $P \leftrightarrow R$ on account of the linkage $Q$ provides between $P$ and $R$. This technique can be extended to link together a longer sequence of biconditionals; the validity of such a chain depends upon the basic form.

This inference rule is used a fair bit in mathematics. In order to prove two sets equal, for instance, you must show that something is an element of one set iff it is an element of the other one. This is often done by showing that an object is an element of the one set iff something else happens, and that this something else happens iff the object is an element of the second set. Stringing these two biconditionals together, then, gives you what is needed.

**Replacement Rules for Conditionals and Biconditionals**

Replacement Rules usually involve more than one connective. They are thus both less simple and less natural than Int-Elim Rules. They are also redundant; we don’t really need to have them in our Deduction System in order for it to be complete. However, they are handy to have because they make our deductions shorter. Here we will consider the Replacement Rules based on the equivalence of simple conditional and biconditional sentences with other forms.

The Replacement Rules **Conditional Equivalence** (Cndl) are often used in order to deduce a conclusion from a conditional sentence $P \rightarrow Q$ in conjunction with other sentences in situations where $MP$ and $MT$ seem inapplicable. They are based upon the equivalences we noted in Section 1.4 when we were justifying the standard truth table for conditional sentences (see also Exercise 1.4-9). The Cndl Replacement Rule is schematized in the following way:

\[
\text{Cndl} \quad \begin{array}{c}
P \rightarrow Q : : \neg (P \land \neg Q) \\
P \rightarrow Q : : \neg P \lor Q
\end{array}
\]

Similarly, the Replacement Rules **Biconditional Equivalence** (Bicndl) are based upon the logical equivalences noted in discussing the semantics of the “iff” connective in Section 1.4 (see also Exercise 1.4-19). They are used most in going from left to right, replacing a biconditional sentence by one in another logical form that may be more easily combined with other sentences.
to deduce a conclusion. But, of course, the right to left direction is also valid. The schema for Biandnl is the following:

\[ P \leftrightarrow Q : : (P \rightarrow Q) \land (Q \rightarrow P) \]

\[ P \leftrightarrow Q : : (P \rightarrow Q) \land (\neg P \rightarrow \neg Q) \]

\[ P \leftrightarrow Q : : (P \land Q) \lor (\neg P \land \neg Q) \]

To illustrate these rules we will show how they can be used (with more effort in this case!) to do the work of BiTrans.

\[ \text{EXAMPLE 1.6 - 6} \]

Show that \( P \leftrightarrow Q, Q \leftrightarrow R \vdash P \leftrightarrow R \) without using BiTrans.

1

\section*{Solution}

The following proof diagram establishes this claim. The biconditionals are taken apart, the separate conditionals linked together appropriately, and the resulting pieces put back together in a biconditional.

\begin{center}
\begin{tabular}{ccc}
   & Prem & Prem \\
1 & \( P \leftrightarrow Q \) & \\
2 & \( Q \leftrightarrow R \) & \\
3 & \((P \rightarrow Q) \land (Q \rightarrow P)\) & Bicndnl 1 \\
4 & \((Q \rightarrow R) \land (R \rightarrow Q)\) & Bicndnl 2 \\
5 & \( P \rightarrow Q \) & Simp 3 \\
6 & \( Q \rightarrow R \) & Simp 4 \\
7 & \( P \rightarrow R \) & HS 5, 6 \\
8 & \( Q \rightarrow P \) & Simp 3 \\
9 & \( R \rightarrow Q \) & Simp 4 \\
10 & \( R \rightarrow P \) & HS 9, 8 \\
11 & \((P \rightarrow R) \land (R \rightarrow P)\) & Conj 7, 10 \\
12 & \( P \rightarrow R \) & Bicndnl 11 \\
\end{tabular}
\end{center}

There is one more standard equivalence for conditional sentences (Contraposition), but since it is often used in the context of deducing a conditional, we will leave it for Section 1.7.

\section*{Biconditional Elimination and Overall Proof Strategy}

While the above rules of inference are rather simple, their crucial importance for the craft of constructing mathematical proofs should not be overlooked or minimized. For example, in attempting to construct a proof of some unfamiliar result, such as

\textit{All widgets that squibble are whatsits},

beginning proof-makers often run up against a mental block right at the outset, not knowing where to start. The best thing you can do when this happens is to look at what you are supposed to conclude and ask both: ‘What does it say?’ and ‘How can I conclude this from what I already know?’ In order to answer these two questions, you will have to engage in both some mathematics and some logic.

On the \textit{logical side}, you will have to be aware of the logical structure of the sentences involved and know what proof strategies are available for deducing sentences like the conclusion from sentences like the premises (the propositions you already know). We discussed this point last lesson; we won’t repeat ourselves here.
On the mathematical side, which is our present focus, you will have to remind yourself of the meaning of the terms involved and of any alternative characterizations they may have been given earlier. You will have to know what widgets and whatsis are and what squibbling involves before you can show that “All widgets that squibble are whatsis”. You have probably already used this approach to study the material in this text: the notions of logical implication and derivation, of consistency and independence, were most likely new to you, so when you were asked to show that a set of sentences implied or proved another sentence or that they were independent, you may have had to review the definition given in the text to see just what you were being asked to do.

In expanding such technical terms, replacing them by an equivalent expression, you are applying BE (or NBE) to the definitions or their alternatives. Doing this as you begin to construct a proof may get you some new propositions that can function as intermediate conclusions, either as an initial step away from the premises, or as a penultimate conclusion. Once you have this, you’re on your way using the Backward-Forward Method to construct a proof. You may still have a lot of work ahead of you, and you may get stumped at times, but at least you’ve overcome the psychological hurdle of getting started and you’ve gained some more ideas about what to try next.

Anytime you run stuck in a proof, whether at the outset or somewhere in the middle, it pays to call a time out, to take some distance to the problem and view your work in more general terms. What exactly are the objects, properties, and relations you are working with? Get beyond the particular widget in front of you and note what type it is a particular instance of. Then ask yourself what properties and relations squibbling widgets have in general. Have you taken full advantage of what you know about them? If not, can you use this knowledge in the particular case at hand? BE (and sometimes MP) permit you to do so, if it helps.

Such a double tactic, thinking both mathematically and logically, becomes increasingly important as you begin to study more abstract mathematics, where the subject matter is no longer the concrete concepts of numbers and shapes that you’ve always thought mathematics deals with. There the definitions of terms will often be quite unfamiliar to you and may seem to make no more sense than the sentence ‘All widgets that squibble are whatsis’. You will thus need to review time and again what it is you are really trying to show (mathematical meaning) and how it might be shown (logical proof strategy). Both aspects are necessary to good proof strategy. Logic without mathematical insight and understanding operates in a fog. But mathematical intuition without logical rigor remains disjointed.

If you follow the above advice and refuse to be daunted by terms that are initially strange, you should have a good start on learning the elements of any field of mathematics. As you proceed, this approach should become second nature to you, so that you no longer need to practice it consciously. You will then have internalized the procedure mathematicians tacitly use all the time when they are working on proofs. By reflecting on proof strategy the way we are doing in this book, you will be developing professional habits that should help you in constructing mathematical proofs in any area of mathematics.

**EXERCISE SET 1.6**

*Problems 1-8: Soundness of Inference Rules*

Show that the following Int-Elim and Replacement Rules are sound by means of extended truth tables.

1. **Modus Ponens (MP):** \( P \rightarrow Q, P \vdash Q. \)
2. **Modus Tollens (MT):** \( P \rightarrow Q, \neg Q \vdash \neg P. \)
3. **Biconditional Elimination (BE):** \( P \leftrightarrow Q, P \vdash Q. \)
4. **Negative Biconditional Elimination (NBE):** \( P \leftrightarrow Q \), \( \neg Q \models \neg P \).

5. **Hypothetical Syllogism (HS):** \( P \rightarrow Q \), \( Q \rightarrow R \models P \rightarrow R \).

6. **Biconditional Transitivity (BiTrans):** \( P \leftrightarrow Q \), \( Q \leftrightarrow R \models P \leftrightarrow R \).

*7. **Conditional Equivalence (Cndnl)**

   a. \( P \rightarrow Q \models \neg (P \land \neg Q) \)
   b. \( P \rightarrow Q \models \neg P \lor Q \)

8. **Biconditional Equivalence (Bicndnl)**

   a. \( P \leftrightarrow Q \models (P \rightarrow Q) \land (Q \rightarrow P) \)
   b. \( P \leftrightarrow Q \models (P \rightarrow Q) \land (\neg P \rightarrow \neg Q) \)
   c. \( P \leftrightarrow Q \models (P \land Q) \lor (\neg P \land \neg Q) \)

9. Show that the following fallacious inferences, cited in this section, are **unsound**; that is, show the argument form to be invalid, in the following two ways:

   i) by assigning truth values to the sentences that make the premises true but the conclusion false,
   
   ii) by giving a counter-argument that is in the same form but that is patently invalid, proceeding from true premises to a false conclusion.

   a. **Affirming the Consequent:** \( P \rightarrow Q \), \( Q \neq P \).
   
   b. **Denying the Antecedent:** \( P \rightarrow Q \), \( \neg P \neq \neg Q \).

10. Formulate the criterion of triangle congruence known by the acronym ASA. Now suppose that you have two triangles \( \triangle ABC \) and \( \triangle A'B'C' \) that have two angles and an included side respectively congruent. What sentential rule of inference is used in concluding via ASA that the triangles themselves are congruent?

**Problems 11-14: Implication and Derivation**

Determine whether the following implications hold by considering possible truth-value assignments for the sentence letters involved. Then show that the derivations given for them are inconclusive. Identify each point where a rule of inference is misused and explain what is wrong with its application.

*11. \( P, (P \rightarrow Q) \rightarrow R \models Q \)?

   1 \hline
   P
   \hline
   Prem

   2 \hline
   (P \rightarrow Q) \rightarrow R
   \hline
   Prem

   3 \hline
   Q
   \hline
   MP 2, 1

12. \( P, (P \rightarrow Q) \rightarrow R \models R \)?

   1 \hline
   P
   \hline
   Prem

   2 \hline
   (P \rightarrow Q) \rightarrow R
   \hline
   Prem

   3 \hline
   R
   \hline
   MP 2, 1

13. \( P \land Q \rightarrow R, \neg R \models \neg Q \)?

   1 \hline
   P \land Q \rightarrow R
   \hline
   Prem

   2 \hline
   \neg R
   \hline
   Prem

   3 \hline
   Q \rightarrow R
   \hline
   Simp 1

   4 \hline
   \neg Q
   \hline
   NBE 3, 2

*14. \( P \rightarrow Q, P \rightarrow R \models P \rightarrow (Q \land R) \)?

   1 \hline
   P \rightarrow Q
   \hline
   Prem

   2 \hline
   P \rightarrow R
   \hline
   Prem

   3 \hline
   \neg P \lor Q
   \hline
   Cndnl 1

   4 \hline
   \neg P \lor R
   \hline
   Cndnl 2

   5 \hline
   (\neg P \land \neg P) \lor (Q \land R)
   \hline
   Conj 3, 4

   6 \hline
   \neg P \lor (Q \land R)
   \hline
   Idem 5

   7 \hline
   P \rightarrow (Q \land R)
   \hline
   Cndnl 6

1.6-9
Problems 15-16: True or False
Are the following statements true or false? Explain your answer.
15. Modus Ponens is the Inference Rule that concludes P from P → Q and Q.
16. Negative Biconditional Elimination is the Inference Rule that concludes ¬Q from P ↔ Q and ¬P.

Problems 17-27: Deductions
Using the rules of inference governing ∧, →, and ↔ that are available so far, construct deductions for the following sequents, putting them into formal proof diagrams with rules of inference cited as reasons for each step.
17. P → (P → Q), P ⊢ P ∧ Q
18. P ∧ Q ⊢ R, P, Q ⊢ R
19. P ⊢ Q, (P → Q) → P ⊢ Q
20. P ∧ (Q → R), ¬R ⊢ P ∧ ¬Q
21. P ⊢ Q, ¬Q ⊢ P ∧ R
22. P ⊢ Q, ¬P ⊢ (R ∧ S), ¬Q ⊢ ¬P ∧ R
23. P ∧ Q ⊢ R, P → S, R ⊢ Q ∧ S
24. P ∧ Q → ¬S, R → (T → S), P, Q, R ⊢ ¬T
25. P → R, Q → R, P ∧ ¬S ⊢ T, T ⊢ R ∧ ¬Q
26. P ⊢ Q, Q → R ∧ S ⊢ ¬P ⊢ (R ∧ S)
27. P → Q, Q → R, R → P ⊢ P ⊢ R

Problems 28-32: Implication and Derivation
Show that the following claims of logical implication are true and then explain why the associated derivation claims gotten by replacing ⊢ with ⊢ cannot be demonstrated at this point in our development of a natural deduction system for SL.
28. P ∧ Q ⊢ P → Q
29. P ⊢ P → P
30. P ⊢ Q → P ∧ Q
31. ¬(P → Q) ⊢ P ∧ ¬Q
32. (P ∨ Q) ∧ ¬Q ⊢ P
33. Interderivability of the Laws of Logic
Show using the rules of inference available so far that given any one of the following laws of logic, the other two can be derived from it.
   i. Law of Identity: P → P
   ii. Law of Non-Contradiction: ¬(P ∧ ¬P)
   iii. Law of Excluded Middle: ¬P ∨ P

Problems 34-40: Negating Definitions
Exercise Set 1.4 asked you to give a number of definitions for various terms. Using them, what statement can you conclude from the following negative information? What rule of inference are you using when you make such a conclusion? [Note: you do not need to further 'simplify' your negations at this stage.]
34. △ABC is not an isosceles triangle.
35. l and m are not perpendicular lines.
36. n is not a composite number.
37. a does not divide b (symbolized by a / b).
38. c is not a zero (root) of a function f.
39. $f$ is not a one-to-one function.
40. $S$ is not a subset of $T$.

**Problems 41 - 48: Proof Strategy**
State what overall proof strategy you would use to get started on a proof of the following theorems, taken from various fields of mathematics. How does BE enter into this process? [Note: you are not expected to be familiar with these results: that’s the whole point of these problems!]

41. The points of intersection of the adjacent trisectors of the angles of a triangle form the vertices of an equilateral triangle.

*42. If $ABCD$ is a Saccheri quadrilateral, then the summit $CD$ is longer than the base $AB$.

*43. Every non–zero finite dimensional inner product space has an orthonormal basis.

44. If $A$ is an invertible matrix, then $A^{-1} = \frac{1}{\text{det}(A)} \text{adj}(A)$.

45. If $G$ is a finite group and $H$ is a subgroup of $G$, then the order of $H$ divides the order of $G$.

46. The set of transcendental numbers is uncountable.

47. Any open covering of a closed, bounded set of real numbers can be reduced to a finite subcovering.

48. Every compact Hausdorff space is normal.

**Problems 49 - 50: Explorations**
The following problems explore aspects of logic related to this section.

49. **Hypothetical Syllogism: Library Exploration**
   a. Use a logic text or a good dictionary or the *Encyclopedia of Philosophy* to look up the following terms connected with the system of Aristotelian or Syllogistic Logic: *syllogism, universal affirmative sentence (A-sentence), Barbara, Hypothetical Syllogism*.
   b. Translate each of the A-sentences of a typical *Barbara* syllogism into conditional sentences. What form of argument results? Comment on the name chosen for this form.

50. **Definitions and Substitution of Biconditionals**
   a. Let $P(\cdot Q \cdot)$ denote any formula of SL that contains an instance of a sentence $Q$, and let $R$ be any sentence of SL. Also, let $P(\cdot R \cdot)$ denote a sentence in which $R$ is substituted for $Q$ in $P(\cdot Q \cdot)$. Carefully explain why the rule of inference that concludes $P(\cdot R \cdot)$ from $P(\cdot Q \cdot)$ and $Q \leftrightarrow R$ is a sound rule.
   b. Thinking of $Q \leftrightarrow R$ as giving a definition for $Q$ and of $P(\cdot Q \cdot)$ as any statement about $Q$, what does this rule tell us about definitions?
4. [No hint.]
7. a. [No hint.]
11. Line 3 is problematic.
14. This deduction has more than one problem.
20. Use Int-Elim Rules for both $\land$ and $\rightarrow$ in this deduction.
21. Start by using $Cndnl$.
22. $BE$ and $MT$ are key rules in this deduction.
26. Rewrite the first premise in order to work with the second one.
27. [No hint.]
28. Explain why we can’t deduce $P \rightarrow Q$ at this stage.
34. In all three arguments, the $Bicndnl$ and $Cndnl$ replacement rules should be used.
35. Use $NBE$.
38. [No hint.]
43. Definitions have $\leftrightarrow$ as the main connective.
44. Definitions have $\leftrightarrow$ as the main connective.
Inference Rules for Sentential Logic

**ELEMENTARY RULES OF INference**

*Premises (Prem)*

A premise may be put down at any line in a deduction

*Reiteration (Reit)*

A sentence may be reiterated at any line in a deduction

**ELIMINATION AND INTRODUCTION RULES FOR ∧**

*Simplification (Simp)*

\[
\begin{array}{c}
P \land Q \\
P \\
Q
\end{array}
\]

*Conjunction (Conf)*

\[
\begin{array}{c}
P \\
Q \\
P \land Q
\end{array}
\]

**ELIMINATION AND INTRODUCTION RULES FOR →**

*Modus Ponens (MP)*

\[
\begin{array}{c}
P \rightarrow Q \\
P \\
Q
\end{array}
\]

*Modus Tollens (MT)*

\[
\begin{array}{c}
P \rightarrow Q \\
\neg Q \\
\neg P \\
\neg P
\end{array}
\]

*Conditional Proof (CP)*

\[
\begin{array}{c}
P \\
Q
\end{array}
\]

*Hypothetical Syllogism (HS)*

\[
\begin{array}{c}
P \rightarrow Q \\
Q \rightarrow R \\
P \rightarrow R
\end{array}
\]

**ELIMINATION AND INTRODUCTION RULES FOR ↔**

*Bicndnl Elim (BE)*

\[
\begin{array}{c}
P \leftrightarrow Q \\
P \\
Q
\end{array}
\]

*Neg Bicndnl Elim (NBE)*

\[
\begin{array}{c}
P \leftrightarrow Q \\
\neg Q \\
\neg P \\
\neg Q
\end{array}
\]

*BiTrans (BT)*

\[
\begin{array}{c}
P \leftrightarrow Q \\
Q \\
P \leftrightarrow R
\end{array}
\]

*Cyclic Bicndnl Int (CBI)*

\[
\begin{array}{c}
P \rightarrow Q \\
Q \rightarrow R \\
R \rightarrow P \\
P \leftrightarrow R
\end{array}
\]

*Bicndnl Int (BI)*

\[
\begin{array}{c}
P \\
Q
\end{array}
\]

*Bicndnl Int, Con (BICOn)*

\[
\begin{array}{c}
P \\
\neg P \\
\neg Q
\end{array}
\]

\[
\begin{array}{c}
P \leftrightarrow Q \\
P \leftrightarrow Q
\end{array}
\]
INTRODUCTION AND ELIMINATION RULES FOR $\neg$

**Negation Introduction (NI)**

\[
\begin{array}{c}
\text{P} \\
\text{Q} \\
\text{\neg Q}
\end{array}
\]

\[
\text{\neg P}
\]

**Negation Elimination (NE)**

\[
\begin{array}{c}
\text{\neg P} \\
\text{Q} \\
\text{\neg Q}
\end{array}
\]

\[
\text{P}
\]

ELIMINATION AND INTRODUCTION RULES FOR $\lor$

**Disjunctive Syllogism (DS)**

\[
\begin{array}{c}
\text{P \lor Q} \\
\text{\neg P}
\end{array}
\]

\[
\text{Q}
\]

**Excluded Middle (LEM)**

\[
\begin{array}{c}
\text{P \lor \neg P}
\end{array}
\]

**Cases**

\[
\begin{array}{c}
\text{P} \\
\text{R}
\end{array}
\]

\[
\begin{array}{c}
\text{Q} \\
\text{R}
\end{array}
\]

**Addition (Add)**

\[
\begin{array}{c}
\text{P}
\end{array}
\]

\[
\begin{array}{c}
\text{Q}
\end{array}
\]

**Either–Or (EO)**

\[
\begin{array}{c}
\text{\neg P}
\end{array}
\]

\[
\begin{array}{c}
\text{\neg Q}
\end{array}
\]

**Rules of Replacement**

**De Morgan’s Rules (DeM)**

\[
\neg (P \land Q) \vdash \neg P \lor \neg Q
\]

\[

\neg (P \lor Q) \vdash \neg P \land \neg Q
\]

**Negative Conditional (Neg Cndnl)**

\[

\neg (P \rightarrow Q) \vdash \neg P \land \neg Q
\]

**Negative Biconditional (Neg BiCndnl)**

\[


\neg (P \leftrightarrow Q) \vdash \neg (P \land Q) \lor (\neg P \land Q)
\]

**Conditional Equiv (Cndnl)**

\[

\text{P} \rightarrow Q \vdash \neg (P \land \neg Q)
\]

\[

\text{P} \rightarrow Q \vdash \neg \neg P \lor Q
\]

**Exportation (Exp)**

\[

\text{P} \rightarrow (Q \rightarrow R) \vdash (P \land Q) \rightarrow R
\]

\[

\text{P} \rightarrow (Q \land R) \vdash (P \lor Q) \land (P \rightarrow R)
\]

**Idempotence (Idem)**

\[

\text{P} \vdash P \land P
\]

\[

\text{P} \vdash P \lor P
\]

**Commutation (Comm)**

\[

P \land Q \vdash Q \land P
\]

\[

P \lor Q \vdash Q \lor P
\]

**Association (Assoc)**

\[

P \land (Q \land R) \vdash (P \land Q) \land R
\]

\[

P \lor (Q \lor R) \vdash (P \lor Q) \lor R
\]

**Double Negation (DN)**

\[
\neg \neg P \vdash P
\]

**Contraposition (Conpsn)**

\[

P \rightarrow Q \vdash \neg Q \rightarrow \neg P
\]

**Biconditional Equiv (Bicndnl)**

\[

P \leftrightarrow Q \vdash (P \land Q) \lor (\neg P \land \neg Q)
\]

\[

P \leftrightarrow Q \vdash (P \land Q) \land (P \lor Q)
\]

\[

P \leftrightarrow Q \vdash (P \land Q) \land (P \lor Q)
\]

**Distribution (Dist)**

\[

P \land (Q \lor R) \vdash (P \lor R) \land (Q \lor R)
\]

\[

(P \lor Q) \land (P \land R) \vdash (P \lor R) \land (Q \lor R)
\]

\[

P \lor (Q \land R) \vdash (P \lor Q) \land (P \lor R)
\]

\[

(P \lor Q) \land R \vdash (P \land R) \land (Q \land R)
\]

\[

P \land (Q \lor R) \vdash (P \land Q) \lor (P \land R)
\]

\[

P \land (Q \land R) \vdash (P \land Q) \lor (P \land R)
\]

\[

P \lor (Q \lor R) \vdash (P \lor Q) \lor (P \lor R)
\]

\[

P \lor (Q \land R) \vdash (P \lor Q) \lor (P \lor R)
\]
1.7 Introduction Rules for *IF-THEN* and *IFF*

So far we have Int-Elim Rules for conjunction and Elim Rules for conditional and biconditional sentences. These are all similar in that they proceed directly in a step-by-step fashion from the premises to conclusions that are based upon them. The rules for introducing conditional and biconditional conclusions are a little different. The *suppositional* arguments they warrant proceed by first making an assumption in addition to the premises that have been given. Based on what has been supposed and on what has been deduced from it, a conclusion is drawn.

Arguments based upon a supposition or a temporary hypothesis occur frequently in mathematics, as well as in ordinary reasoning; in fact, such arguments typify mathematical reasoning. If you listen to mathematicians as they develop proofs, you’ll hear words like ‘suppose’ or ‘let’ come up time and time again. These terms indicate that a suppositional argument is about to begin. Suppositional reasoning is the lifeblood of mathematical proof.

**Natural Deduction Systems and Suppositional Inference Rules**

In order to capture the underlying logic of mathematical reasoning in our deduction system, we will admit several suppositional rules of inference to our Deduction System. As we will see, mirroring actual reasoning patterns is what makes this system so appealing to practicing mathematicians and gives it a decided advantage in usefulness over other deduction systems.

We noted in Section 5 that the deduction system we are developing is a Jaśkowski-Fitch style Natural Deduction System. One of its main distinguishing features in comparison with the older, Frege-Hilbert style deduction systems is the inclusion of and emphasis upon *suppositional rules of inference*.

Though the use of suppositional argumentation captures the way proofs are ordinarily made, suppositional rules of inference are slightly more complicated than the other rules. It may take you a little longer to become thoroughly familiar with the way they work, but once you have mastered this, they will make your formal deductions simpler, more structured, more flexible, and more enjoyable. Moreover, constructing suppositional arguments will help you to learn useful proof strategy and to reason the same way arguments are structured in theoretical mathematics and computer science courses.

A suppositional argument temporarily assumes a supposition as a working premise or an auxiliary hypothesis in order to prove a desired result. Suppositions are only temporarily assumed because the final conclusion does not have the supposition among its list of premises. A supposition is indeed a premise of a *subordinate proof* or an *inner subproof*, but at the conclusion of such a subproof (occasionally two such subproofs), a conclusion is drawn back in the main body of the proof based *not* upon the *suppositions*, but upon the *entire subproof* (or subproofs) that the supposition helped to spawn. Once its mission is complete, a supposition no longer functions as a premise: it is honorably *discharged*.

The inference drawn is thus not dependent upon one or two sentences of specific logical forms, as has been the case thus far with the Int-Elim Rules we’ve given for “and”, “if-then”, and “iff”. The conclusion depends instead on what has been *deduced from* the supposition posited; it follows from the whole subordinate demonstration.

To schematize arguments containing subproofs based on temporary suppositions, we will indent each subproof in our formal deductions (a practice analogous to programming conventions for subroutines). We will underline the supposition being assumed, since it functions as a working premise in the subproof, and we will mention why we are making that assumption in the right-hand column. Then, once the appropriate final conclusion has been drawn in the subproof, we will go on to draw a *related conclusion* in the *main part* of the proof, citing the appropriate rule of inference applied to the entire subordinate argument as the reason for the
inference. The proof schema will thus look roughly like the following, where for the sake of simplicity we are assuming only one subproof:

\[
\begin{array}{c}
\begin{align*}
j \quad \cdots \quad \text{Inf Rule X h, i} \\
k \quad \cdots \quad \text{Spsn for Spsnal Inf Rule Y} \\
m \quad \cdots \\
n \quad \cdots \quad \text{Spsnal Inf Rule Y k-m}
\end{align*}
\end{array}
\]

Since suppositions are not premises supplied at the start of a proof, you may wonder where they come from. Well, from anywhere. You are completely free to assume any supposition you wish for the sake of argument in a subproof. That’s the beauty of subproofs; they allow you to explore “what if this were the case?” Of course, not any old sentence will turn out to be a profitable assumption. Moreover, the proposition or information that you are permitted to carry back into the main body of the proof at the end of the subproof depends in an essential way upon what you assumed and what followed from your supposition. Such conclusions are carefully governed by suppositional rules of inference; suppositional rules must be sound, just like other rules of inference.

So how do you decide then what supposition might be appropriate to try? You use the Backward Method of Proof Analysis! You ask yourself, “What sort of sentence am I trying to prove here? What rule of inference will allow me to get such a sentence?” If the rule you need is a suppositional rule, the rule itself will help you decide what supposition to choose in order to get what you want.

For example, suppose you want to prove the conditional sentence ‘if a function \( f \) is differentiable, then \( f \) is continuous’. The way you would typically do this is to start by supposing that \( f \) is a differentiable function. You then use what you know about differentiability and try to show that \( f \) must be continuous. This being done, you would consider the conditional result proved.

At this point our description is still rather abstract and fuzzy, but the nature and use of suppositional arguments will become clearer once you see a few specific inference rules applied to concrete examples and begin making deductions on your own. By the time the entire Natural Deduction System is in place for Sentential Logic, you will have become very familiar with a number of these rules. In this section you will learn about three suppositional rules of inference, one for proving or introducing conditional sentences, and two for introducing biconditional sentences. In the following two sections we will study suppositional rules related to negation and disjunction.

We will also consider a few other rules of inference here that are not suppositional rules. Some of these are Replacement Rules associated with conditional sentences; and one is primarily a bookkeeping rule related to the construction of suppositional arguments in general. We will begin with a brief discussion of this rule, the Rule of Reiteration, since it is needed in a great percentage of suppositional proofs.

### Suppositional Proofs and the Rule of Reiteration

Subordinate proofs inside a main proof proceed much as the main proof itself. Given the premise (the supposition), you argue further, concluding sentences that follow from it by sound rules of inference.
As you continue, however, you are not restricted merely to the single supposition you’ve made; you should also be able to use all the results you’ve already gotten up to that point in the proof. How you access these lines in a subproof is largely a matter of organizational style. You could just refer back to whatever line you need, citing it in the reason column. But we will adopt a tidier approach in our formal deductions: we will repeat any line we want to use inside the subproof before using it. In this way each subproof becomes a self-contained proof of its own. The rule of inference that permits us to do this is the Rule of Reiteration (Reit). Given a sentence already deduced in the main argument, we can repeat this sentence later in the proof or in any subproof of the argument. This amounts to a license to move any sentence down and in, from some given line in the proof to a later line in any of its subproofs. You can also repeat a sentence within the same proof-level, citing Reit there, too, but this is unnecessary for the most part, since the earlier line can already be cited.

The inference schema for Reit is the following:

\[
\text{Reit} \quad \begin{array}{c|c}
  P & P \\
\end{array}
\]

Though sentences may be moved further into a proof, the opposite is not permitted. None of the sentences that are obtained within a subproof may be simply moved out to the main part of the proof or to another subproof. For, being based upon a supposition, which is not a premise of the overall argument, such sentences cannot be exported to the main proof. Such a rule of inference would be wildly unsound; it would allow you to prove anything whatsoever, since there is no restriction on what can be supposed as the working premise of a subproof.

\[\text{EXAMPLE 1.7 - 1}\]

What is wrong with the following argument, purporting to show that \( P \rightarrow Q \mid \vdash Q \)?

1. \( P \rightarrow Q \) Prem
2. \( P \) Spsn
3. \( P \rightarrow Q \) Reit 1
4. \( Q \) MP 3, 2
5. \( Q \) Reit 4

\[\text{Solution}\]

The problem here is the invalid Reiteration of \( Q \) from line 4 onto line 5. Everything else is strictly according to code, including the Reiteration in line 3.

\[\text{Introduction Rule for IF-THEN: Conditional Proof}\]

In proving sentences of the form \( P \rightarrow Q \) the most natural approach in the world (for a mathematician, anyway) is to suppose for the sake of argument that \( P \) actually is the case and then to deduce \( Q \) from it. Having done this, you conclude that \( P \rightarrow Q \) holds.

Such a procedure is tacitly followed in mathematics all the time in proving conditional statements. We had one example above (differentiability implies continuity). To give another one, suppose we want to prove the proposition ‘if \( \triangle ABC \) is isosceles, then its base angles \( \angle A \) and \( \angle B \) are congruent’. We would take an isosceles triangle, label the vertices \( A, B, \) and \( C, \) and
then show that the base angles are congruent. In other words, we would take the antecedent ‘\(\triangle ABC\) is isosceles’ as a premise for our argument, treating it as ‘given’, and then derive the consequent ‘base angles \(\angle A\) and \(\angle B\) are congruent’ as the conclusion. Some authors even formulate the proposition’s statement in multi-sentence form, making the antecedent appear to be a premise at the outset, saying something like ‘Suppose \(\triangle ABC\) is isosceles. Then the base angles \(\angle A\) and \(\angle B\) are congruent’. In any case, having demonstrated that the antecedent proves the consequent, the conditional result is considered to be established without further ado. It should be clear that the procedure described here is the same as that described above: the sentence ‘if \(P\) then \(Q\)’ is taken as proved because \(P\) proves \(Q\).

Such an inference is made according to the rule of Conditional Proof (CP). On the basis of having a subproof of \(Q\) from \(P\) (and not on the basis of a fixed number of sentences of certain logical types) you can assert \(P \rightarrow Q\). The conclusion \(P \rightarrow Q\) is a conditional statement and in no way depends upon \(P\) actually being the case (on \(\triangle ABC\) being isosceles); hence \(P\) is not a premise of the argument that establishes the proposition \(P \rightarrow Q\). It does, of course, function as a premise of the subproof establishing \(Q\) (the base angles \(\angle A\) and \(\angle B\) are congruent), but not of the conditional sentence finally proved. The supposition \(P\) has been plowed into the conditional conclusion, as it were, and so has been discharged.

Conditional Proof is the prototype for all suppositional rules, and was introduced independently by Jaśkowski and Gentzen in the mid-1930’s. The proof schema for this rule is given below. Note that the entire subproof of \(Q\) from \(P\) is placed above the doubleunderline. Only after you have demonstrated that \(Q\) is a deductive consequence of \(P\) (possibly requiring many lines and a great deal of work on your part) are you permitted to conclude \(P \rightarrow Q\).

\[
\text{CP} \\
\begin{array}{c}
P \\
\hline \\
Q \\
\hline \\
P \rightarrow Q
\end{array}
\]

To illustrate the use of Conditional Proof in a formal derivation, we will deduce the contrapositive of a conditional sentence from the sentence itself. We will shortly adopt a replacement rule (Contraposition) for doing this, but the following example shows that this is redundant.

\[\text{EXAMPLE 1.7 - 2}\]

Show that \(P \rightarrow Q \vdash \neg Q \rightarrow \neg P\).

\textbf{Solution}

Using the Backward Method of proof analysis, note that we want to conclude \(\neg Q \rightarrow \neg P\), a conditional sentence. The strategy for accomplishing this is to use CP (see below).

We thus suppose \(\neg Q\) as a temporary premise and work to prove \(\neg P\).

To construct this inner argument, we use the given premise, \(P \rightarrow Q\), first reiterating it to make it available in the subproof.

The sentences \(P \rightarrow Q\) and \(\neg Q\) are then combined via \(\text{MT}\) to give what we want.

Once the subproof is complete, we are warranted in concluding \(\neg Q \rightarrow \neg P\) in the main part of the proof.

The proof diagram for this deduction is as follows:

| 1 | \(P \rightarrow Q\) | Prem |
| 2 | \(\neg Q\) | Spsn for CP |
| 3 | \(P \rightarrow Q\) | Reit 1 |
| 4 | \(\neg P\) | MT 3, 2 |
| 5 | \(\neg Q \rightarrow \neg P\) | CP 2-4 |
There may be times when a formal deduction using \( CP \) will seem contrived or unnatural to you. This is not the fault of \( CP \), but is a feature of the truth-functional connective \( \rightarrow \), which can be used to connect any two sentences \( P \) and \( Q \) regardless of their logical connection, making \( P \rightarrow Q \) true whenever \( Q \) is true or \( P \) is false. Thus, in showing that \( Q \vdash P \rightarrow Q \) in the next example, \( P \) plays no active role whatsoever in deducing \( Q \) in the subproof, since no premise is required. \( Q \) being the case, it doesn’t matter what \( P \) is; \( P \rightarrow Q \) is also the case. We still consider this to be a proof of \( Q \) from \( P \) via \( CP \). Such proofs are not typical of genuine applications of \( CP \) in mathematical or other concrete settings, where the antecedent and consequent are logically connected, but you may run across them from time to time in a formal setting. Nothing is endangered by such arguments, just as nothing is threatened by having nonsensical conditional sentences. You merely learn to live with them as a byproduct of having a truth-functional connective that mirrors logical implication inside SL.

**EXAMPLE 1.7 - 3**

Show that if a sentence is unconditionally true, then it is also conditionally true, given any condition whatsoever; that is, show that \( Q \vdash P \rightarrow Q \).

**Solution**

The following proof diagram establishes the claim.

\[
\begin{array}{c|c}
1 & Q & \text{Prem} \\
2 & P & \text{Spsn for CP} \\
3 & Q & \text{Reit 1} \\
4 & P \rightarrow Q & \text{CP 2-3} \\
\end{array}
\]

The next example illustrates the use of \( CP \) in another, more natural argument, and it also shows that Hypothetical Syllogism is redundant, given \( CP \) and \( MP \). We will keep HS in our Natural Deduction System, however, so that we don’t have to go through a lengthy argument like this every time we want to chain two conditionals together.

**EXAMPLE 1.7 - 4**

Show that \( P \rightarrow Q, Q \rightarrow R \vdash P \rightarrow R \), without using HS.

**Solution**

The following proof diagram establishes the claim.

\[
\begin{array}{c|c}
1 & P \rightarrow Q & \text{Prem} \\
2 & Q \rightarrow R & \text{Prem} \\
3 & P & \text{Spsn for CP} \\
4 & P \rightarrow Q & \text{Reit 1} \\
5 & Q & \text{MP 4, 3} \\
6 & Q \rightarrow R & \text{Reit 2} \\
7 & R & \text{MP 6, 5} \\
8 & P \rightarrow R & \text{CP 3-7} \\
\end{array}
\]

Let’s now briefly consider the soundness of \( CP \). Why can we legitimately conclude \( P \rightarrow Q \) at the end of a \( CP \) argument? Well, if we can prove \( Q \) on the basis of supposing \( P \) to be the case, then \( Q \) must also follow from \( P \).* And if \( Q \) logically follows from \( P \), then the conditional

---

* Assuming the use of sound inference rules, of course. Since the subproof may itself involve suppositional inferences, a rigorous argument that \( CP \) and other suppositional rules are sound is more involved. We will not pursue this here.
sentence $P \rightarrow Q$ must be logically true according to the *Implication Theorem* (Section 1.4). On the basis of deducing $Q$ from $P$ in a subproof, therefore, we have the right to assert the conditional sentence $P \rightarrow Q$; its truth is established.*

**Additional Replacement Rules for Conditionals**

In Section 1.6 we gave the most basic Replacement Rules for conditional and biconditional sentences, ones based on alternative formulations of those sentences in terms of other connectives. Here we will add two more Replacement Rules for conditional sentences: *Contraposition* ($Conpsn$) and *Exportation* ($Exp$). Although, as we will see, these Replacement Rules may be applied more in one direction than another and are usually applied to full sentences, they are two-directional Replacement Rules since the two sentences are logically equivalent: either side can replace the other in an argument. And, as equivalents, these sentences can also be substituted for component parts of sentences.

The first rule is based on the contrapositive equivalent of a conditional sentence. It can be used to prove a conditional statement $P \rightarrow Q$ by proving its equivalent $\neg Q \rightarrow \neg P$. This is an extremely valuable proof technique; it is used often in mathematical proofs. It is really no more complicated than $CP$, though when you see it being used in practice you may wonder at first why a negation $\neg Q$ is being assumed in order to prove the positive conclusion $Q$ from the positive premise $P$. We will have more to say about how this rule works in Section 1.8, when we compare it with a closely related procedure involving negations. *Contraposition* is schematized as follows:

$$Conpsn \quad P \rightarrow Q \quad :\quad \neg Q \rightarrow \neg P$$

*Contraposition* has a counterpart that holds for biconditionals: $P \leftrightarrow Q \quad :\quad \neg Q \leftrightarrow \neg P$. Since this is less useful for making proofs, we will not adopt it as one of our inference rules.

Another replacement rule is known as *Exportation* ($Exp$). It consists of two rules that tell how certain compound sentences involving $\land$ and $\rightarrow$ can be expanded or contracted. They are schematized as follows:

$$Exp \quad P \rightarrow (Q \rightarrow R) \quad :\quad (P \land Q) \rightarrow R$$
$$P \rightarrow (Q \land R) \quad :\quad (P \rightarrow Q) \land (P \rightarrow R)$$

The first *Exportation* rule often remains implicit or unused because mathematicians and others tend to formulate complex conditionals in terms of conjunction from the start instead of using a nested conditional. A sentence of the form $P \rightarrow (Q \rightarrow R)$ seems easier to understand in its equivalent form $(P \land Q) \rightarrow R$; it is clearer there just what is given and what needs to be deduced from it (via $CP$). Compound conditionals are rather awkward to state and understand due to the different levels of conditions generated by the multiple ‘if-then’ connectives. Moreover, without parentheses it’s often unclear just how the clauses are to be separated, and since $P \rightarrow (Q \rightarrow R)$ is not logically equivalent to $(P \rightarrow Q) \rightarrow R$ (see Exercise 48), using a conjunctive form avoids any potential confusion about which conditional sentence is really intended.

Nevertheless, there occasionally are informal statements of mathematical propositions in which conditionals are piled up. To clarify how the conditionals are linked together, this is often done by means of a trailing ‘provided’ clause. The sentence ‘if $ab = ac$, then $b = c$, provided

* Frege-Hilbert style deduction systems, which lack *Conditional Proof* (as well as other suppositional rules of inference), make up for it by proving a meta-theorem known as the *Deduction Theorem*, which says $P \vdash Q$ iff $\vdash P \rightarrow Q$, the deductive counterpart to our *Implication Theorem*. This theorem was proved independently by Tarski and Herbrand, two twentieth-century logicians. We prefer to incorporate this form of reasoning within our system of logic instead.
"a \neq 0" is a good example of this. We can rephrase this as ‘if \(a \neq 0\), then if \(ab = ac, b = c\)', which is of the form \(P \rightarrow (Q \rightarrow R)\). The way in which such a proposition is usually proved, though, is by implicitly assuming the first Exportation rule to get started: “suppose \(a \neq 0\) and \(ab = ac\).” The formal justification for this maneuver is the top Exp formula, proceeding from left to right. This initial conclusion must be made explicit in a formal deduction. Proceeding right to left via Exp occurs even less frequently in mathematical proofs, if at all, but it, too, would be legitimized by this replacement rule.

The second exportation rule is also used implicitly in mathematics. It is most often applied in order to deduce a sentence in the form of the left hand sentence, \(P \rightarrow (Q \land R)\). In an informal proof you would merely show that each of the conjuncts of the right hand side, \(P \rightarrow Q\) and \(P \rightarrow R\), hold. You would then assume the proof was finished without further conjoining these conditionals or citing Exp. The rules Conj and Exp, however, are what validate such a procedure, and so should be cited in formal arguments. Reading this bottom rule from left to right shows that it can be considered a distributive law of sorts (\(\rightarrow\) distributes over \(\land\)), but we will not call it that to avoid confusing it with the other distributive statements for \(\rightarrow\), which are not sound rules of inference (see Exercises 49–51).

\[\text{EXAMPLE 1.7.5}\]
Show that \(\neg P \lor Q, \neg P \rightarrow R \vdash \neg Q \rightarrow (R \land \neg P)\).

\[\text{Solution}\]
The following proof diagram demonstrates this claim.

| 1 | \(\neg P \lor Q\) | Prem |
| 2 | \(\neg P \rightarrow R\) | Prem |
| 3 | \(P \rightarrow Q\) | Cndnl 1 |
| 4 | \(\neg Q \rightarrow \neg P\) | Conpsn 3 |
| 5 | \(\neg Q \rightarrow R\) | HS 4, 2 |
| 6 | \((\neg Q \rightarrow R) \land (\neg Q \rightarrow \neg P)\) | Conj 5, 4 |
| 7 | \(\neg Q \rightarrow (R \land \neg P)\) | Exp 6 |

\[\text{Introduction Rule for IFF: Biconditional Introduction}\]

We can use what we’ve learned about proving conditional sentences to see how to prove biconditional sentences. Since \(P \leftrightarrow Q\) is logically equivalent to \((P \rightarrow Q) \land (Q \rightarrow P)\), we can prove such a sentence by first proving \(P \rightarrow Q\) and then proving \(Q \rightarrow P\). Each of these can be done in turn by means of Conditional Proof. Since we are not really interested here in establishing each conditional separately, but both conjointly (the biconditional), we will omit drawing a conditional conclusion at the end of each subproof and just use the two subproofs together as the reason for inferring the biconditional. This is a proof strategy we have used informally a couple of times already: we used it to prove the Implication Theorem, for instance, even before we began our discussion of deductions.

This approach, called Biconditional Introduction (BI), is schematized in the following way:
The remarks made with respect to CP apply here, too. In order to conclude the biconditional \( P \leftrightarrow Q \), you must first develop two subproofs, one based upon the supposition \( P \), the other based upon the supposition \( Q \). No other sentences besides the conclusions \( Q \) and \( P \) have been indicated in the subproof schemas, but any given argument in which \( BI \) is used may contain many intermediate conclusions as well.

In certain cases, the proof you construct for the second subproof for \( BI \) will be almost exactly parallel to the one you’ve just done for the first subproof. When this occurs in an informal setting, you will probably read something like, “The other direction follows similarly” as the conclusion to the proof. It occasionally happens, too, that instead of using the proof as a model, the result itself just proved (the conditional whose converse you still need to deduce) can be used to make the second subproof in short order. Example 1.5-2 (commutativity of conjunction) was like this. Keep your eyes open for these tactics and you’ll see them.

The following examples illustrate how to construct formal deductions using \( BI \). The first example is a fairly simple one and is related to the \( Bicndnl \) replacement rule (see Section 1.6). The second one illustrates that a subproof can itself contain subproofs. Suppositional argument-forms thus add complexity to our deductions, but they also introduce a modular deductive structure to help us comprehend this complexity. The second example also shows how suppositional rules of inference can be used to prove results that have no premises, something you may not have thought possible (proving something from nothing). Such results are ones that are also valid with no premises—that is, tautologies.

\[ \text{EXAMPLE 1.7 - 6} \]
Show that \( P \to Q, \quad \neg P \to \neg Q \vdash P \leftrightarrow Q \) via \( BI \), without using \( Bicndnl \) or \( Conpsn \).

**Solution**

The following proof diagram establishes the claim.

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>( P \to Q )</td>
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<tr>
<td>2</td>
<td>( \neg P \to \neg Q )</td>
</tr>
<tr>
<td>3</td>
<td>( P )</td>
</tr>
<tr>
<td>4</td>
<td>( P \to Q )</td>
</tr>
<tr>
<td>5</td>
<td>( Q )</td>
</tr>
<tr>
<td>6</td>
<td>( Q )</td>
</tr>
<tr>
<td>7</td>
<td>( \neg P \to \neg Q )</td>
</tr>
<tr>
<td>8</td>
<td>( \neg \neg P )</td>
</tr>
<tr>
<td>9</td>
<td>( P )</td>
</tr>
<tr>
<td>10</td>
<td>( P \leftrightarrow Q )</td>
</tr>
</tbody>
</table>

\[ \text{EXAMPLE 1.7 - 7} \]
Show that \( \vdash [P \to (Q \land R)] \leftrightarrow [(P \to Q) \land (P \to R)] \) via \( BI \), without using \( Exp \).

**Solution**

We will begin by analyzing what is needed, using the Backward-Forward Method. The deduction looks horrendously long at first glance, but if you go over it slowly, using the proof analysis as a guideline and paying particular attention to the overall structure of the argument, you’ll find it’s not difficult to follow. After you’ve been doing suppositional proofs for a while, this one will seem pretty straight-forward and quite natural, even if long.

In order to deduce the biconditional we’re given using \( BI \), we will generate two subproofs: \( (P \to Q) \land (P \to R) \) from \( P \to (Q \land R) \) (lines 1-12); and conversely, \( P \to (Q \land R) \) from \( (P \to Q) \land (P \to R) \) (lines 13-21).
For the first subproof: to deduce the conjunction \((P \rightarrow Q) \land (P \rightarrow R)\) (line 12), we will first deduce each conjunct separately (lines 6, 11).

In order to deduce these two conjuncts, each of which is a conditional sentence, we will use CP. We’ll look at a proof of the first one, \(P \rightarrow Q\) (lines 2-6).

CP permits us to suppose \(P\) (line 2); then we must prove \(Q\).

Since we are given \(P \rightarrow (Q \land R)\) as a premise and \(P\) as our supposition, MP allows us to detach \(Q \land R\) (lines 2-4).

Using Simp on this conjunction gives us \(Q\) (line 5), which is what we want.

Having gotten \(Q\) on the assumption that \(P\) is the case, we can now conclude \(P \rightarrow Q\) via CP in the main part of the subproof (line 6).

The rest of the first subproof proceeds in the same way (lines 7-11).

The second subproof (lines 13-21) is even easier than the first, containing only one second-level subproof. It begins with a conjunction and derives a conditional sentence by means of CP (lines 14-21).

Thus, given both subproofs, we conclude the biconditional (line 22):

\[
[P \rightarrow (Q \land R)] \leftrightarrow [(P \rightarrow Q) \land (P \rightarrow R)].
\]

Here is the proof diagram for this example.

1. \(P \rightarrow (Q \land R)\)  
   - Spsn 1 for BI

2. \(P\)  
   - Spsn for CP

3. \(P \rightarrow (Q \land R)\)  
   - Reit 1

4. \(Q \land R\)  
   - MP 3, 2

5. \(Q\)  
   - Simp 4

6. \(P \rightarrow Q\)  
   - CP 2-5

7. \(P\)  
   - Spsn for CP

8. \(P \rightarrow (Q \land R)\)  
   - Reit 1

9. \(Q \land R\)  
   - MP 8, 7

10. \(R\)  
    - Simp 9

11. \(P \rightarrow R\)  
    - CP 7-10

12. \((P \rightarrow Q) \land (P \rightarrow R)\)  
    - Conj 6, 11

13. \((P \rightarrow Q) \land (P \rightarrow R)\)  
    - Spsn 2 for BI

14. \(P\)  
    - Spsn for CP

15. \((P \rightarrow Q) \land (P \rightarrow R)\)  
    - Reit 13

16. \(P \rightarrow Q\)  
    - Simp 15

17. \(Q\)  
    - MP 16, 14

18. \(P \rightarrow R\)  
    - Simp 15

19. \(R\)  
    - MP 18, 14

20. \(Q \land R\)  
    - Conj 17, 19

21. \(P \rightarrow (Q \land R)\)  
    - CP 14-20

22. \([P \rightarrow (Q \land R)] \leftrightarrow [(P \rightarrow Q) \land (P \rightarrow R)]\)  
    - BI 1-12, 13-21

Note that the deduction would have been slightly shorter if from supposition \(P\) (line 2) we had shown in a single subproof both \(Q\) (line 5) and \(R\) (line 10); lines 7-9 simply repeat lines 2-4. We could then draw both conditional conclusions (lines 6 and 11) in the main
part of the proof. While this is not the official way $CP$ is applied, nothing can be said against taking such a shortcut in informal argumentation.

**Introduction Rule for IFF: Contrapositive Form of BI**

Since a conditional sentence is logically equivalent to its contrapositive, we can get another introduction rule for biconditional sentences out of the schema for $BI$ simply by replacing either of the subproofs there by a subproof of its contrapositive. The resulting inference rule will likewise be sound. Thus, if you first work toward proving $P \rightarrow Q$ (though, as in $BI$, without explicitly drawing that conclusion in your argument) and then work to conclude $\neg P \rightarrow \neg Q$ in place of its equivalent $Q \rightarrow P$, you can legitimately conclude $P \leftrightarrow Q$. This was the upshot of Example 6 above.

This leads us to the following rule of inference, which we will call *Biconditional Introduction, Contrapositive Form* ($BICon$). The schematic representation of this rule of inference is as follows:

```
BICon

\[ \begin{array}{c}
P \\
\hline
Q \\
\hline
\neg P \\
\hline
\neg Q \\
\hline
P \leftrightarrow Q
\end{array} \]
```

It is this form of *Biconditional Introduction* that you will often meet in mathematical proofs. Such a proof usually starts out by saying “Suppose that $P$ is the case”, and then shows that $Q$ must also be the case. That having been done, the proof continues by saying “Now suppose that $P$ is not the case”, and then shows that $Q$ is not the case either. When that is done, the proof is considered finished. (Informal mathematical proofs generally do not repeat the original proposition, but it is understood without being said that the biconditional proposition has been proved.) Having shown that $Q$ can be derived from $P$ and that $\neg Q$ similarly follows from $\neg P$ (naturally, in conjunction with any premises assumed in the proposition), the desired biconditional conclusion immediately follows according to *Biconditional Introduction* in its *Contrapositive* form.

Proofs that use $BICon$ proceed pretty much the same way as those using $BI$. Since we gave an extended example for $BI$, we will omit giving one for $BICon$.

**Proving Propositions Equivalent: BiTrans and Cyclic BI**

*Biconditional Introduction* is based upon the equivalence $P \leftrightarrow Q \vdash (P \rightarrow Q) \land (Q \rightarrow P)$. This equivalence also appears in the most fundamental of the *Bicondnl* Replacement Rules. However, mathematicians often want to prove that more than two sentences are equivalent to one another. As we mentioned in Section 1.4, biconditional sentences often come up in the context of alternative definitions or characterizations of a concept. When this is done, the different options should be shown to be logically equivalent, so that any one of them can be used whenever the occasion arises. Taking $P_1, \ldots, P_n$ to denote the various characterizations, it must be shown that any two characterizations $P_i$ and $P_j$ are equivalent in the context of the theory. Alternatively, according to the *Generalized Equivalence Theorem*, it must be shown that $P_i \leftrightarrow P_j$ holds in the theory under consideration.
Whenever a series of biconditionals like this needs to be demonstrated, they can naturally be proved individually using BI or BICon, but this may not be the most economical way to proceed. If this is nevertheless done, a version of Hypothetical Syllogism for biconditionals, which we will call Biconditional Transitivity (BiTrans), can be used to shorten the process a bit. This rule is schematized as follows:

\[
\begin{array}{c}
\text{BiTrans} \\
\hline
P_1 \leftrightarrow P_2 \\
P_2 \leftrightarrow P_3 \\
\hline
P_1 \leftrightarrow P_3
\end{array}
\]

An alternative to this is to prove the equivalences in one fell swoop for the entire group by constructing proofs for a series of related conditional sentences, more or less in cyclical fashion. Beginning with sentence \(P_1\), you first prove \(P_1 \rightarrow P_2\), then \(P_2 \rightarrow P_3\), and so on, finishing up with \(P_n \rightarrow P_1\) to complete the cycle. Because these sentences form a closed chain of conditionals, it is clear that any sentence can be used to prove any other (use HS repeatedly), and hence that each associated biconditional sentence also holds. The various sentences are thus equivalent to one another in the context of the theory being studied.

There are lots of variations on this proof procedure that you may run across. Sometimes it turns out to be easier to construct several interlinked cycles rather than a single large cycle, and so the deductions will be done that way instead (see Exercise 36). Regardless of how it is done, however, such cycling around usually saves quite a bit of leg work over showing each biconditional sentence separately.

The rule of inference that underlies this cyclic proof procedure is an extension of BI: if \(P_1 \rightarrow P_2\), \(P_2 \rightarrow P_3\), and \(P_3 \rightarrow P_1\), then \(P_1 \leftrightarrow P_j\) for any \(P_1\) and \(P_j\) (cf. Exercise 1.6-27). It is tempting to name such an inference BI-Cycling, but since that would prompt additional groans and this is a serious mathematics text, we’ll instead call it Cyclic Biconditional Introduction (CycBI). The schema for this rule is as follows.

\[
\begin{array}{c}
\text{CycBI} \\
\hline
P_1 \rightarrow P_2 \\
P_2 \rightarrow P_3 \\
P_3 \rightarrow P_1 \\
\hline
P_1 \leftrightarrow P_j
\end{array}
\]

To illustrate how and where this rule is used in mathematics, consider the notion of a square matrix being invertible; i.e., having an inverse under matrix multiplication. This concept, which is usually introduced early in an elementary linear algebra course, turns out to have close connections to several other ideas (linear independence, being one-to-one, etc.). A blockbuster theorem can thus be formulated that presents all the various characterizations as equivalent. The proof of this theorem proceeds by some form of Cyclic Biconditional Introduction, showing how each of the ideas implies the others. Similar sorts of characterizations occur in most other areas of mathematics; in fact, the importance of a concept can be gauged to some extent by how many other concepts it is equivalent to.

**Suppositional Rules of Inference and Global Proof Strategy**

You’ve now been introduced to three suppositional Introduction Rules, CP, BI, and BI-Con, and you’ve seen how proofs are constructed using such rules. Suppositional rules provide global logical structure to proofs by breaking them into subproofs. Constructing such proofs is analogous to creating structured programs in computer science by means of a top-down,
stepwise refinement procedure. Using the Backward-Forward Method of Proof Analysis to determine what needs proving and then constructing subproofs for each of these key propositions, you can break up a long and complex proof into more manageable components. At this point we are mainly using the Backward Method to get started on our proofs, but as we proceed we will find suppositional rules that will benefit from the Forward Method as well.

Suppositional arguments provide you with a better overview of what is going on in a proof. Each subproof is there for a particular reason and contributes its part to the whole. Paying attention to these subproofs and the conclusions they generate, you will have a better grasp of how such a proof is proceeding overall, and you will be better equipped to construct a deduction of the conclusion from the premises on your own. The proof diagrams we have adopted help you visualize this logical structure by separating levels of subproofs with indented vertical sidelines. Proofs will no longer seem to be a long string of propositions whose order must be memorized; they take shape in such a way that the main points become apparent from the diagram.

At first you may have a little difficulty working suppositional proofs and making proof diagrams, but with practice you should be able to do most of them with ease, regardless of their length. While you will not put proofs into diagram form in other contexts, learning how to do suppositional proofs in this way is extremely beneficial. Since most mathematical propositions involve a conditional statement in some way, CP gets used over and over again in their proofs. Moreover, by being aware of how subproofs work, you will realize that you can’t always take a result demonstrated earlier in the proof and simply use it again; it all depends upon where it was proved in the argument and how it functioned earlier in the proof.

**EXERCISE SET 1.7**

**Problems 1-6: Soundness of Inference Rules**
Show the soundness of the following Int-Elim and Replacement Rules (not all of which are in the text) by means of extended truth tables.

1. Conditional Reflexivity: \( \models P \rightarrow P \)
2. Biconditional Reflexivity: \( \models P \leftrightarrow P \)
3. Biconditional Symmetry: \( P \leftrightarrow Q \models Q \leftrightarrow P \)
4. Contraposition (Conpsn): \( P \rightarrow Q \models \neg Q \rightarrow \neg P \)
5. Exportation (Exp)
   a. \( P \rightarrow (Q \rightarrow R) \models (P \land Q) \rightarrow R \)
   b. \( P \rightarrow (Q \land R) \models (P \rightarrow Q) \land (P \rightarrow R) \)
6. Biconditional Transitivity (BiTrans) and Cyclic Biconditional Introduction (CycBI)
   a. \( P \leftrightarrow Q, Q \leftrightarrow R \models P \leftrightarrow R \)
   b. \( P \rightarrow Q, Q \rightarrow R, R \rightarrow P \models P \leftrightarrow R \)

**Problems 7-10: Completing Deductions**
Fill in the reasons for the following deductions.

*7. \( P \rightarrow Q \models P \rightarrow (R \rightarrow Q) \)

\[
\begin{array}{l}
| P \rightarrow Q | \\
| P \rightarrow Q | \\
| P \rightarrow Q | \\
| P \rightarrow Q | \\
| P \rightarrow Q | \\
| P \rightarrow Q | \\
| P \rightarrow Q | \\
| P \rightarrow Q | \\
\end{array}
\]
Problems 11-13: Logical Implication and Conclusive Deductions

Determine whether the following claims of logical implication are true. Then determine whether the deductions are conclusive. Carefully point out where rules of inference are being used incorrectly.

**Problem 11:**

\[ P \land Q \rightarrow R, P \rightarrow Q \vdash P \rightarrow R \]

1. \[ P \land Q \rightarrow R \] (Prem)
2. \[ P \rightarrow Q \] (Spsn for CP)
3. \[ P \] (Spsn for CP)
4. \[ Q \] (Reit 2)
5. \[ P \land Q \] (Conj 3, 4)
6. \[ P \land Q \rightarrow R \] (Reit 1)
7. \[ R \] (MP 6, 5)
8. \[ P \rightarrow R \] (CP 2-7)

**Problem 12:**

\[ (P \rightarrow P \land Q) \leftrightarrow (P \rightarrow Q) \]

1. \[ P \rightarrow P \land Q \]
2. \[ P \]
3. \[ P \rightarrow P \land Q \]
4. \[ P \land Q \]
5. \[ Q \]
6. \[ P \rightarrow Q \]
7. \[ P \rightarrow Q \]
8. \[ P \rightarrow P \land Q \]
9. \[ P \rightarrow Q \]
10. \[ Q \]
11. \[ P \land Q \]
12. \[ P \rightarrow P \land Q \]
13. \[ (P \rightarrow P \land Q) \leftrightarrow (P \rightarrow Q) \]

**Problem 13:**

\[ (P \rightarrow P \land Q) \leftrightarrow (P \rightarrow Q) \]

1. \[ P \rightarrow P \land Q \]
2. \[ P \]
3. \[ P \rightarrow P \land Q \]
4. \[ P \land Q \]
5. \[ Q \]
6. \[ P \rightarrow Q \]
7. \[ P \rightarrow Q \]
8. \[ P \rightarrow P \land Q \]
9. \[ P \rightarrow Q \]
10. \[ Q \]
11. \[ P \land Q \]
12. \[ P \rightarrow P \land Q \]
13. \[ (P \rightarrow P \land Q) \leftrightarrow (P \rightarrow Q) \]
*12. \( P \lor \neg Q, P \land Q \rightarrow \neg R \models \neg P \rightarrow R \)

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<tr>
<td>1</td>
<td>( P \lor \neg Q )</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>( P \land Q \rightarrow \neg R )</td>
<td>Prem</td>
</tr>
<tr>
<td>3</td>
<td>( P \rightarrow Q )</td>
<td>Cndnl 1</td>
</tr>
<tr>
<td>4</td>
<td>( P )</td>
<td>Spsn for CP</td>
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<tr>
<td>5</td>
<td>( P \rightarrow Q )</td>
<td>Reit 3</td>
</tr>
<tr>
<td>6</td>
<td>( Q )</td>
<td>MP 5, 4</td>
</tr>
<tr>
<td>7</td>
<td>( P \land Q )</td>
<td>Conj 4, 6</td>
</tr>
<tr>
<td>8</td>
<td>( P \land Q \rightarrow \neg R )</td>
<td>Reit 2</td>
</tr>
<tr>
<td>9</td>
<td>( \neg R )</td>
<td>MP 8, 7</td>
</tr>
<tr>
<td>10</td>
<td>( P \rightarrow \neg R )</td>
<td>CP 4–9</td>
</tr>
<tr>
<td>11</td>
<td>( \neg P \rightarrow R )</td>
<td>Conpsn 10</td>
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</table>

*13. \( P \rightarrow Q \land R, \neg Q, P \rightarrow \neg R \models \neg P \)

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<tbody>
<tr>
<td>1</td>
<td>( P \leftrightarrow Q \land R )</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>( \neg Q )</td>
<td>Prem</td>
</tr>
<tr>
<td>3</td>
<td>( P \rightarrow \neg R )</td>
<td>Prem</td>
</tr>
<tr>
<td>4</td>
<td>( P )</td>
<td>Spsn for Conj</td>
</tr>
<tr>
<td>5</td>
<td>( P \rightarrow \neg R )</td>
<td>Reit 3</td>
</tr>
<tr>
<td>6</td>
<td>( \neg R )</td>
<td>MP 5, 4</td>
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<tr>
<td>7</td>
<td>( \neg Q )</td>
<td>Reit 2</td>
</tr>
<tr>
<td>8</td>
<td>( \neg Q \land \neg R )</td>
<td>Conj 7, 6</td>
</tr>
<tr>
<td>9</td>
<td>( \neg P )</td>
<td>NBE 1, 8</td>
</tr>
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**Problems 14-18: True or False**

*Are the following statements true or false? Explain your answer.*

14. The rule *Exportation* can only be used for replacing whole sentences.

15. Suppositional arguments are quite common in mathematical proofs.

16. Suppositional arguments introduce modular structure into formal deductions.

17. The rule *Reiteration* can be used any time you want to reiterate a previous line somewhere in the proof.

18. *Cyclic Biconditional Introduction* concludes a *biconditional* based on a cycle of *conditional* statements.

**Problems 19-36: Deductions**

*Using the rules of inference that are available so far, construct deductions for the following sequents, putting them into formal proof diagrams with rules of inference cited as reasons for each step.*

19. \( P \land Q \vdash P \rightarrow Q \)

20. \( P \rightarrow Q \vdash P \land R \rightarrow Q \)

21. \( P \rightarrow Q \land R \vdash P \rightarrow Q \)

22. \( P \vdash (P \rightarrow Q) \rightarrow Q \)

23. \( Q \rightarrow R \vdash P \land Q \rightarrow P \land R \)

24. \( P \rightarrow Q, R \rightarrow S \vdash P \land R \rightarrow Q \land S \)

25. \( P \rightarrow Q \vdash (Q \rightarrow P) \rightarrow (P \rightarrow Q) \)

26. \( P \rightarrow Q \vdash (Q \rightarrow R) \rightarrow (P \rightarrow R) \)

27. \( (P \rightarrow Q) \rightarrow R \vdash Q \rightarrow R \)

28. \( (P \rightarrow Q) \leftrightarrow R \vdash P \rightarrow (Q \leftrightarrow R) \)

29. \( P \rightarrow (Q \rightarrow R), Q \vdash P \rightarrow R \)
30. $P \rightarrow Q \vdash [P \rightarrow (Q \rightarrow R)] \rightarrow (P \rightarrow R)$
31. $\vdash P \rightarrow P$
32. $\vdash P \rightarrow (Q \rightarrow P)$
33. $\vdash P \rightarrow [(P \rightarrow Q) \rightarrow Q]$
34. $P \rightarrow R, Q \rightarrow \neg R \vdash P \rightarrow \neg Q$
35. $P \rightarrow Q, Q \rightarrow R, R \rightarrow S, S \rightarrow P \vdash P \leftrightarrow P \leftrightarrow S$
*36. $P \rightarrow Q, Q \rightarrow R, R \rightarrow P, S \rightarrow Q, R \rightarrow S \vdash P \leftrightarrow P \leftrightarrow S.$

37. Show that any instance of the Law of Excluded Middle, $P \lor \neg P$, can be derived as a theorem of logic (i.e., from no premises) via CP and Cntrl. Thus Cntrl is not acceptable to intuitionist logicians/mathematicians.

Problems 38-46: Interderivability

Using only Int-Elim Rules, show that the following sentences are interderivable by giving a deduction of each sentence from the other one.
38. $P \rightarrow (P \rightarrow Q) \vdash P \rightarrow Q$
39. $P \rightarrow (Q \rightarrow R) \vdash (P \land Q) \rightarrow R$
40. $P \rightarrow (Q \rightarrow R) \vdash Q \rightarrow (P \rightarrow R)$
41. $P \rightarrow (Q \land R) \vdash (P \rightarrow Q) \land (P \rightarrow R)$
42. $P \rightarrow (Q \rightarrow R) \vdash (P \rightarrow Q) \rightarrow (P \rightarrow R)$
43. $P \rightarrow Q \vdash Q \rightarrow P$
44. $P \rightarrow Q \vdash (P \rightarrow Q) \land (Q \rightarrow P)$
45. $P \rightarrow Q \vdash (P \rightarrow Q) \land (\neg P \rightarrow \neg Q)$
46. $P \rightarrow Q \vdash \neg P \leftrightarrow \neg Q$

Problems 47-52: Bogus Replacement Rules

Show, as indicated, that the following bogus replacement rules do not hold. Then tell whether either of the two sentences implies the other one of the pair and construct a deduction to show it if it does.
47. Idempotence for $\rightarrow$: $P \rightarrow P :/\vdash P$.
48. Commutation for $\rightarrow$: $P \rightarrow Q :/\vdash Q \rightarrow P$.
49. Association for $\rightarrow$: $P \rightarrow (Q \rightarrow R) :/\vdash (P \rightarrow Q) \rightarrow R$.
50. Left Distribution of $\land$ over $\rightarrow$: $P \land (Q \rightarrow R) :/\vdash (P \land Q) \rightarrow (P \land R)$.
51. Right Distribution of $\land$ over $\rightarrow$: $(P \rightarrow Q) \land R :/\vdash (P \land R) \rightarrow (Q \land R)$.
52. Right Distribution of $\rightarrow$ over $\land$: $(P \land Q) \rightarrow R :/\vdash (P \rightarrow R) \land (Q \rightarrow R)$.

Problems 53-55: Formulating Mathematical Theorems

Symbolize each of the following mathematical theorems using the notation of SL. Then write out its logical equivalent, using Exportation. Put your final answer back into mathematical English.

*53. Intermediate Value Theorem

If a function $f$ is continuous on a closed interval $[a, b]$, then if $d$ is any number between $f(a)$ and $f(b)$, there exists a number $c$ between $a$ and $b$ such that $f(c) = d$.

54. Ptolemy’s Theorem

If $ABCD$ is a quadrilateral inscribed in circle, then if $AC$ and $BD$ are the diagonals of $ABCD$, $AC \cdot BD = AB \cdot CD + BC \cdot AD$.

55. The $\sqrt[4]{T}$ is continuous at a real number $a$ for a function $f$ that is continuous at $a$, provided $f(a) > 0$ when $n$ is an even number.
\*56. Infinite Series and Convergence

Given two infinite positive-term series $\sum a_n$ and $\sum b_n$ with $a_n \leq b_n$ for all $n$, one calculus textbook proceeded to prove the following two propositions independently of one another:

(a) If $\sum b_n$ converges, then $\sum a_n$ also converges;
(b) If $\sum a_n$ diverges, then $\sum b_n$ also diverges.

What rule of inference could the author have used to shorten his work? Explain.

**Problems 57 - 67: Proof Strategy**

Determine the logical forms of the following propositions. Then tell which rule(s) of inference you would expect to use in proving them. You do not need to attempt the proofs, but you should outline some of the key steps (from a logical point of view) that might occur in the proofs and explain what proof strategies could be used in connection with them.

\*57. If integers $a$ and $b$ are odd, then $a \cdot b$ is odd.

58. An integer $a$ is even iff $a^2$ is even.

59. An even number times an odd number is even.

60. If $q$ is a rational number and $i$ is an irrational number, then $q \cdot i$ is irrational.

61. If $a \mid b$ and $b \mid c$ for integers $a$, $b$, and $c$, then $a \mid c$.

62. $\triangle ABC$ is a right triangle iff $a^2 + b^2 = c^2$.

63. If $ABCD$ is a rectangle, then diagonals $AC$ and $BD$ are congruent and bisect one another.

64. The system of equations $ax + by = e$ and $cx + dy = f$ has a unique solution pair of real numbers $(x, y)$ iff $ad - be \neq 0$.

65. The series $\sum a_n$ converges iff each associated series $\sum ca_n$ converges for $c \neq 0$.

66. A function $f$ is invertible iff $f$ is one-to-one and $f$ is onto.

\*67. If a set $S$ is countably infinite, then $T$ is countably infinite iff $T$ is equinumerous with $S$.

**Problems 68 - 70: Proofs**

Prove the following propositions in as much detail as you can, identifying all rules of inference with which you are familiar. Use some form of two column proof, reserving the second column for rules of inference (when you know them). Assume that the numbers in each of these exercises are positive integers, unless otherwise noted.

\*68. If $a$ and $b$ are both odd, then $a \cdot b$ is odd.

Definition needed: a number $n$ is odd iff $n = 2k + 1$ for some number $k$.

69. $a$ is even iff $a^2$ is even.

Definition needed: a number $n$ is even iff $n = 2k$ for some number $k$.

70. If $a \mid b$ and $b \mid c$, then $a \mid c$.

Definition needed: $k \mid n$ iff $n = m \cdot k$ for some number $m$. 
HINTS TO STARRED EXERCISES 1.7

7. [No hint.]
10. [No hint.]
12. The proof starts off badly, but that’s not the only problem here.
13. More than one line is problematic in the deduction, but that doesn’t mean anything about whether the premises logically imply the conclusion.
15. [No hint.]
17. [No hint.]
24. Design your deduction by focusing on the conclusion: suppose $P \land R$ and show how $Q \land S$ follows.
27. This one’s a bit tricky. Start by supposing $Q$, as expected. In order to use your premise, though, you need $P \rightarrow Q$, so first prove that, using an inner subproof.
36. Use $CycBI$ here.
53. This sentence is in the form $P \rightarrow (Q \rightarrow R)$.
56. Saying a series diverges is the negation of saying it converges.
57. This sentence is in the form $P \land Q \rightarrow R$.
67. This sentence is in the form $P \rightarrow (Q \leftrightarrow R)$.
68. Build on Problem 57: suppose that both $a$ and $b$ are odd; flesh out what these mean via the definition; and then show that $a \cdot b$ is odd, using basic algebra to simplify the product.
1.8 Proof by Contradiction: Rules for NOT

We now have inference rules for three sentential connectives: ∧, →, and ↔. We have yet to discuss the rules for ¬ and ∨. This will be our focus in the next two sections.

We will first consider arguments that involve negations. This section will take up both Introduction and Elimination Rules for ¬. Because such proofs are often called indirect proofs, we will begin with a general description of what these are. Then we will discuss and illustrate our two main negation rules. To flesh out our discussion of such proofs, we will introduce some Replacement Rules for simplifying compound negations and illustrate their use. Finally, we will come back to finish our opening analysis of the nature and merits of indirect proofs over against direct proofs.

Direct vs. Indirect Proofs

Proofs have traditionally been classified as “direct” or “indirect”. In both cases premises are used to establish a conclusion, but the methods used are quite different.

The term ‘indirect proof’ is a negative one, meaning “not a direct proof”. Its meaning thus depends upon that of ‘direct proof’. Pursuing this, we might suppose as a first approximation that a direct proof is one whose conclusion follows directly from the premises. This may turn out to be circular description, though, since we need to know what ‘follow directly’ means. It might be thought that direct proofs should exclude suppositional reasoning, which proceeds somewhat obliquely via suppositions and so doesn’t go directly from the premises alone. This is problematic, however, for some suppositional proofs, such as CP, are traditionally classified as direct proofs and are sometimes even taken as their prototype. Even though it is difficult to pin down a precise meaning for direct proof, we will see in the end that there is a sense in which these proofs are more direct than indirect ones.

Another characterization is suggested by an alternative phrase. An older term for direct proof is ‘ostensive proof’, whose etymology seems to imply that a direct proof is one that clearly demonstrates its result. But this poses some perplexing questions. Doesn’t an indirect proof do the same? Are indirect proofs inferior to direct ones? Again, we will see that there is a sense in which direct proofs reveal the logical connection between premises and conclusion more clearly than indirect ones, but the rather fuzzy notion of “clarity” is difficult to make into the main criterion for distinguishing direct from indirect proofs. What is clear to one person may be unclear to someone else.

Let’s take a slightly different tack. Maybe a direct proof is more directed than an indirect proof; equivalently, an indirect proof is more unguided than a direct one. Could this be the case? For instance, maybe direct proofs make more use of the Backward Method (that’s what ordinarily directs proof construction in mathematics) than the indirect ones. As you will see, there is something to this point, too. But is there any way this can be made more precise so that we can use it to distinguish between the two types of proof? Will it help us to formulate or recognize rules of inference that govern indirect proof?

Puzzling over the distinction between direct and indirect proofs, you may start to wonder whether it might not be easier to characterize direct and indirect proofs in the reverse order, first defining ‘indirect proof’ and then calling a ‘direct proof’ any proof that is not indirect. This may not be very useful for characterizing direct proofs, since it will lump together all sorts of different proofs that can be further distinguished, but at least in this way the term ‘indirect proof’ will refer to an important, clearly delineated type of proof. This reverse approach is, in fact, the one we will take.
**Indirect Proof: Proof by Contradiction**

Other terms for “indirect proof”, as we will use it, are *Proof by Contradiction* and *Proof by Reductio ad Absurdum*. The basic proof strategy here, as these terms indicate, is to somehow use a contradiction or an absurdity in order to prove the desired result. More precisely, given a set of premises and a conclusion, we attempt to establish the conclusion on the basis of the premises by showing that accepting the opposite conclusion leads to a patent absurdity, that is, to a contradiction. Thus, if by assuming the logical opposite of the conclusion as a temporary hypothesis, we are able to deduce two contradictory results, we may then conclude from this impossibility that the supposition cannot hold and that the desired conclusion is the case instead.

*Proof by Contradiction* occurs in many places in mathematics. In fact, it is one of the main tools used by mathematicians to prove certain sorts of propositions. *Proof by Contradiction* is not a modern maneuver; it is almost as old as systematic deductive argument itself. Philosophers as far back as Zeno are known to have used it to argue the ridiculousness of their opponents’ positions. *Reductio ad absurdum* arguments are used all the time in debates whenever someone wants to refute an alternative position to his or her own. Aristotle discussed it explicitly in his works and told how it had been used by Greek mathematicians to establish a momentous result, the incommensurability of the side and diagonal of a square. Given a square, it is impossible (even in theory) to find a unit of measure, no matter how small, that can exactly measure both a diagonal and a side using a whole number of units. The modern version of this proposition asserts that \(\sqrt{2}\) (the ratio of the diagonal to the side) is irrational. The classic *Proof by Contradiction* for this result goes as follows:

**THEOREM: Existence of Irrational Numbers**

\(\sqrt{2}\) is Irrational.

**Proof:**

Suppose \(\sqrt{2}\) is rational.

Then \(\sqrt{2}\) can be expressed as a fraction \(m/n\) of integers; assume without loss of generality that \(m/n\) is in fully reduced form – all fractions can be put into such a form.

Squaring both sides of \(\sqrt{2} = m/n\), we obtain \(2 = m^2/n^2\).

\((\ast)\) Clearing of fractions, we get \(2n^2 = m^2\).

Thus \(m^2\) is even.

But then \(m\) itself must be even; otherwise \(m^2\) would be odd.

So \(m = 2k\) for some integer \(k\).

Thus \(m^2 = (2k)^2 = 4k^2\).

Substituting this in the equation of line \((\ast)\) yields \(2n^2 = 4k^2\).

Thus \(n^2 = 2k^2\), which means \(n^2\) is even.

But then \(n\) must be even, too.

But if both \(m\) and \(n\) are even, \(m/n\) is not in reduced form.

This contradicts our earlier statement; and so the supposition that \(\sqrt{2}\) is rational is absurd.

Hence \(\sqrt{2}\) must be irrational. ■

In this proof we have demonstrated that a negative sentence (‘\(\sqrt{2}\) is not rational’) is the case by showing that its contradictory (‘\(\sqrt{2}\) is rational’) is absurd. You might think that the argument only shows that \(m/n\) wasn’t in reduced form; yet after canceling the common factor of 2, the exact same argument will apply to the new fraction, leading to the absurdity that such factors can be forever canceled; in other words, that it is impossible to put a fraction in reduced form, which contradicts what is known.

Such a deduction uses the rule of inference known as *Negation Introduction*, since a negation is introduced as the conclusion. *Negation Elimination* proceeds similarly; it concludes a positive sentence on the basis of its negation leading to a contradiction. Together these two rules of
inference constitute *Proof by Contradiction*. We will now discuss and schematize these two rules of inference.

**Negation Introduction and Negation Elimination**

*Negation Introduction* (NI) can be used whenever we wish to deduce a negative sentence \( \neg P \). Rather than trying to prove \( \neg P \) directly from the premises, we assume its opposite \( P \) and then, in a subproof headed by this supposition, derive both a sentence \( Q \) and its contradictory opposite \( \neg Q \). Once this has been done, on the basis of that subproof, we are permitted to infer \( \neg P \) as a conclusion in the main part of the proof. Schematically we have the following:

\[
\begin{array}{c}
\text{NI} \\
\hline
P \\
Q \\
\neg Q \\
\hline
\neg P
\end{array}
\]

It should be emphasized, as we have done before, that while our schema may make it appear that \( Q \) and \( \neg Q \) follow directly after the supposition \( P \), we only mean to indicate that both \( Q \) and \( \neg Q \) must be deduced in a subproof based upon \( P \) (naturally, in conjunction with any premises or previously proved sentences). In using this rule of inference in a particular proof, you may have many intermediate steps in your subproof, and \( Q \) and \( \neg Q \) may be separated by quite a few lines of argument. Nevertheless, if you can deduce both \( Q \) and \( \neg Q \) from \( P \) somewhere within the subproof, NI allows you to conclude \( \neg P \) back in the main part of the proof.

*Negation Elimination* (NE) is the rule of inference to use when we wish to use contradiction to deduce any sentence \( P \) that is not a negation. Rather than trying to prove \( P \) purely on the basis of the premises we’re given, we assume its negation \( \neg P \) as a temporary assumption and then, using it along with the given premises, we proceed to derive both a sentence \( Q \) and its contradictory \( \neg Q \) in a single subproof. On the basis of this subproof, we are then permitted to infer \( P \) as a conclusion to the proof. Schematically we have the following:

\[
\begin{array}{c}
\text{NE} \\
\hline
\neg P \\
Q \\
\neg Q \\
\hline
P
\end{array}
\]

*Negation Introduction* and *Negation Elimination* are both sound rules of inference. Their soundness can be demonstrated, assuming the soundness of CP, by showing the two equivalences \( P \rightarrow (Q \land \neg Q) \vdash \neg P \) and \( \neg P \rightarrow (Q \land \neg Q) \vdash P \) (see Exercise 5).

These two forms of *Proof by Contradiction* don’t seem to be all that different—in order to deduce what is wanted, each proceeds from a supposition opposed to the original premise and arrives finally at a contradiction. Yet there are some important differences, which lead to their being accepted differently by certain philosophically minded segments of the mathematical and logical communities.
Negation Introduction, on the one hand, is a rule of inference accepted by all mathematicians and logicians. If a sentence $P$ leads to an absurdity, then it cannot be the case and must be denied; that is, its negation $\neg P$ must hold. Logical consistency (the Law of Non-Contradiction), it seems, is recognized as normative for deductive reasoning by practically everyone. If a theory is found to contain a logical absurdity, this is cause for expending some effort to rid the system of the source of such a contradiction. If the basic intuitions of a field of study lead to conflicting results, something is awry with them and they should be modified.

On the other hand, a small minority of people oppose Negation Elimination. While a contradiction generated by a supposition $\neg P$ indicates to everyone that it should be rejected, what it is you can accept is still at issue. According to intuitionist or constructivist mathematicians and logicians, all you can legitimately conclude is what $NI$ gives; namely, $\neg \neg P$. You cannot conclude the positive statement $P$, according to them, because the Law of Excluded Middle cannot be universally accepted. The double negative $\neg \neg P$ is thus taken to be a weaker statement than the related positive statement $P$.

Our attitude will be a qualified acceptance of $NE$. For the vast majority of mathematicians and logicians, $NE$ is seen as a permissible rule of inference. Since $P \models \neg \neg P$ in our system of SL, there is absolutely no reason for rejecting $NE$: we will incorporate it into our Natural Deduction System. However, you may wish to restrict its use to cases where it really does seem necessary, where no other alternative seems to work. There seem to be some good reasons why it should be used sparingly. We will say a bit more about this and about the limitations of using Proof by Contradiction in general at the end of this section.

### Deductions Involving NI and NE

We will now illustrate the two rules of inference $NI$ and $NE$ by working a few examples. We will begin with a proof of the Law of Non-Contradiction ($LNC$). Our Natural Deduction System is set up to do proofs largely without the foundational apparatus of logical axioms, but since tautologies like $LNC$ are logical truths, we will be able to construct an argument for it (as we have done before, in Example 1.7-5) that is completely premise-free. Indeed, such deductions are often the best way to show that a proposition is a tautology.

#### EXAMPLE 1.8 - 1

Prove the Law of Non-Contradiction: $\vdash \neg(P \land \neg P)$.

**Solution**

Let’s first analyze what is needed. Since our conclusion is $\neg(P \land \neg P)$, with main connective $\neg$, our strategy will be to introduce the negation; i.e., to use $NI$.

*Negation Introduction* requires that we suppose the associated positive formula $P \land \neg P$ and then attempt to derive two contradictory sentences.

Since $P \land \neg P$ is a conjunction, all we can do is simplify it. But this is all we need: this generates a contradiction.

The proof thus goes as follows:

<table>
<thead>
<tr>
<th>1</th>
<th>$P \land \neg P$</th>
<th>Spsn for NI</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>$P$</td>
<td>Simp 1</td>
</tr>
<tr>
<td>3</td>
<td>$\neg P$</td>
<td>Simp 1</td>
</tr>
<tr>
<td>4</td>
<td>$\neg(P \land \neg P)$</td>
<td>NI 1-3</td>
</tr>
</tbody>
</table>

We can also use these rules to show that a double negation $\neg \neg P$ can be derived from the positive sentence $P$, and conversely. The former is intuitionistically valid, as we noted above, and will be left as an exercise (Exercise 17). The more controversial direction will be treated in the next example. It illustrates the use of $NE$. 

1.8-4
EXAMPLE 1.8 - 2
Prove one direction of Double Negation: \(\neg\neg P \vdash P\).

Solution
The following proof diagram establishes the claim.

1 \(\neg\neg P\) Prem
2 \(\neg P\) Spsn for NE
3 \(\neg P\) Reit 2
4 \(\neg\neg P\) Reit 1
5 \(P\) NE 2-4

As a corollary to the last two examples, we will be able to prove the Law of Excluded Middle, once we have De Morgan’s Rule available for simplifying negated conjunctions: \(\vdash P \lor \neg P\). We will leave this as an exercise (see Exercise 33).

The following examples have valid-argument forms related to negation and conjunction that are rather simple and might be made into rules of inference having independent status. However, since they do not introduce or eliminate a single connective, we will leave them out of our official list of inference rules. The first of the two is closely related (via one of De Morgan’s Rules; see below) to another rule of inference, known as Disjunctive Syllogism, which we will state in Section 1.9 in connection with “or” elimination. And the other one is easily derived when it is needed, so not much is lost by omitting them both from our deduction system.

EXAMPLE 1.8 - 3
Show that \(P, \neg(P \land Q) \vdash \neg Q\).

Solution
The following proof diagram establishes the claim. A similar argument shows that \(Q, \neg(P \land Q) \vdash \neg P\) (see Exercise 18).

1 \(P\) Prem
2 \(\neg(P \land Q)\) Prem
3 \(Q\) Spsn for NI
4 \(P\) Reit 1
5 \(P \land Q\) Conj 4, 3
6 \(\neg(P \land Q)\) Reit 2
7 \(\neg Q\) NI 3-6

EXAMPLE 1.8 - 4
Show that \(\neg Q \vdash \neg(P \land Q)\) (roughly, a partial converse of the last example).

Solution
The following proof diagram establishes the claim.

1 \(\neg Q\) Prem
2 \(P \land Q\) Spsn for NI
3 \(Q\) Simp 2
4 \(\neg Q\) Reit 1
5 \(\neg(P \land Q)\) NI 2-4
Replacement Rules for Simplifying Negations: NOT, AND, and OR

One of the complexities connected with applying NE is the fact that the negation being supposed usually needs to be simplified in order to proceed further with the argument. The following Replacement Rules enable us to overcome this obstacle. They permit us to replace compound negations with easier to use equivalent forms.

The Replacement Rule Double Negation (DN) was already mentioned above. It warrants us to conclude a positive statement \( P \) based upon its double negation \( \neg\neg P \). This is the direction intuitionist logicians reject in general (see, however, Exercise 33). Double Negation also proceeds in the other, intuitionistically valid, direction. It warrants concluding \( \neg\neg P \) from \( P \). The schema for DN is the following:

\[
DN \quad \neg\neg P :: P
\]

De Morgan’s Rules (DeM) are based upon the equivalences mentioned in Section 1.3. You can use them to operate on a conjunction or disjunction by negation, getting the right hand side; you can also use them to simplify complex disjunctions and conjunctions of negations, obtaining the left hand side. These formulas exhibit a strict duality: in both of them, you negate a compound sentence by negating the individual parts and interchanging \( \land \) with \( \lor \) to get the new principal connective. These rules are very handy and should become second nature to you after you have used them a bit. They are schematized in the following way:

\[
DeM
\]

\[
\neg(P \land Q) :: \neg P \lor \neg Q
\]

\[
\neg(P \lor Q) :: \neg P \land \neg Q
\]

At this point we have not studied disjunctions, so if we were to replace a negated conjunction with its equivalent disjunction (the first form of DeM) in a Proof by Contradiction, we would not be able to proceed further with an immediate inference. After the Int-Elim Rules of Section 1.9 are available, this deficiency will be gone, and we will be able to proceed further with any negation’s equivalent. However, we can already do the work of these rules in a round about way using Proof by Contradiction and DeM, as the following example illustrates.

\[\text{EXAMPLE 1.8 - 5}\]

Show that the following elimination claim for \( \lor \) holds: \( P \lor Q, \neg P \vdash Q \).

\[\text{Solution}\]

We will prove this result by Contradiction, leaving a direct proof via Cndnl as an exercise (see Exercise 26).

<p>| | |</p>
<table>
<thead>
<tr>
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<tbody>
<tr>
<td>1</td>
<td>P \lor Q</td>
</tr>
<tr>
<td>2</td>
<td>\neg P</td>
</tr>
<tr>
<td>3</td>
<td>\neg Q</td>
</tr>
<tr>
<td>4</td>
<td>\neg P</td>
</tr>
<tr>
<td>5</td>
<td>\neg P \land \neg Q</td>
</tr>
<tr>
<td>6</td>
<td>\neg(P \lor Q)</td>
</tr>
<tr>
<td>7</td>
<td>P \lor Q</td>
</tr>
<tr>
<td>8</td>
<td>Q</td>
</tr>
</tbody>
</table>
Replacement Rules for Simplifying Negations: IF-THEN and IFF

Negating conditional and biconditional sentences can be done in stages. First of all, we can replace the inside conditional or biconditional sentence with an equivalent expressed in terms of \(\neg\), \(\land\), and \(\lor\). Then we can simplify its negation, using \(DN\) and \(DeM\) as needed. For example, using the Replacement Rule \(Bicndnl\), \(\neg(P \leftrightarrow Q)\) is equivalent to \(\neg((P \land Q) \lor (\neg P \land \neg Q))\); applying \(DeM\), this in turn is equivalent to \(\neg(P \land Q) \land \neg(\neg P \land \neg Q)\), which can then be taken apart by \(Simp\) and worked with further. It can also be shown to lead eventually to its equivalent \((P \land \neg Q) \lor (\neg P \land Q)\). We can shorten this whole process by adopting a Replacement Rule that does it in one step. \(Negative\ \textit{Biconditional}\) is schematized as follows.

\[
\neg(P \leftrightarrow Q) : : (P \land \neg Q) \lor (\neg P \land Q)
\]

This rule will probably not come into play very often, for two reasons. In the first place, negating biconditionals doesn’t seem to arise very often in a natural setting, so there is little call to use it. But in the second place, its conclusion can be obtained in short order from \(DeM\) and the next Replacement Rule, \(Negative\ \textit{Conditional}\), after first putting the biconditional into its most natural equivalent form, \((P \to Q) \land (Q \to P)\). So while we will officially admit \(Neg\ \textit{Biconditional}\) into our Natural Deduction System, \(Negative\ \textit{Conditional}\) is the more valuable rule.

In order to simplify a negated conditional in the same way as above, we would proceed through the following equivalents, applying the first form of \(Cndnl\) and \(DN\):

\[
\neg(P \to Q) \vdash (P \land \neg Q) \lor (\neg P \land Q)
\]

Rather than go through this sequence of steps each time, we will adopt the Replacement Rule \(Negative\ \textit{Conditional}\) (\(Neg\ \textit{Cndnl}\)), which infers \(P \land \neg Q\) directly from \(\neg(P \to Q)\) without an intermediate step. Schematically, we have the following:

\[
\neg(P \to Q) : : P \land \neg Q
\]

Using \(Neg\ \textit{Cndnl}\), we can prove a conditional sentence by \(NE\) in the following way. We first suppose \(\neg(P \to Q)\) to get the subproof going; then in order to proceed, we use \(Neg\ \textit{Cndnl}\) to conclude that \(P \land \neg Q\) must be the case. By \(Simp\), this immediately gives us two new sentences, \(P\) and \(\neg Q\), which can in turn be used in our subproof to help generate a contradiction. A formal proof diagram taking this approach would look as follows:

\[
\begin{array}{c}
\neg(P \to Q) \quad \text{Spsn for NE} \\
P \land \neg Q \quad \text{Neg Cndnl} \\
P \quad \text{Simp} \\
\neg Q \quad \text{Simp}
\end{array}
\]

This procedure explains why informal mathematical arguments that use \textit{Proof by Contradiction} on a conditional statement \(P \to Q\) begin by saying, “Suppose \(P\) is the case and \(Q\) is not.” They are merely developing an argument form based upon \(Neg\ \textit{Cndnl}\).

The next example illustrates this use of \(Neg\ \textit{Cndnl}\) more fully in a formal deduction.

\begin{itemize}
  \item \textbf{EXAMPLE 1.8-6}
  \begin{itemize}
    \item Show that \(P \land Q \to R, Q \leftrightarrow \neg R \vdash P \to R\).
  \end{itemize}
\end{itemize}
**Solution**

The following proof diagram establishes this result.

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<tbody>
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<td>(Q \leftrightarrow \neg R)</td>
<td>Prem</td>
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<td>(\neg R)</td>
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<td>7</td>
<td>(Q \leftrightarrow \neg R)</td>
<td>Reit 2</td>
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<td>8</td>
<td>(Q)</td>
<td>BE 7, 6</td>
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<td>9</td>
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<td>10</td>
<td>(P \land Q \rightarrow R)</td>
<td>Reit 1</td>
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<td>11</td>
<td>(R)</td>
<td>MP 10, 9</td>
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<tr>
<td>12</td>
<td>(P \rightarrow R)</td>
<td>NE 3-11</td>
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Neg Cndnl is primarily used in Proof by Contradiction settings. It might therefore be formulated as a one-directional rule of inference, proceeding from \(\neg (P \rightarrow Q)\) to \(P \land \neg Q\). However, since these two formulas are logically equivalent, we have formulated it as a more versatile two-directional Rule of Replacement.

**Contraposition and Proof by Contradiction**

Contraposition is sometimes (wrongly) considered a form of Contradiction, even by some mathematics educators, so we will briefly analyze and compare the two methods in order to properly distinguish them.

To prove a sentence like \(P \rightarrow Q\), we have several methods available. The most common one uses CP: assuming \(P\), we prove \(Q\). This direct method remains our first line of attack. But now we also have Proof by Contradiction available. This can be used in one of two ways, but they both amount to the same thing. Either we can assume \(P\) and then try to prove \(Q\) by NE, or we can try to prove the entire sentence \(P \rightarrow Q\) by NE, using Neg Cndnl. Either way, we start with \(P\) and \(\neg Q\) and have to come up with a contradiction.

Contraposition gives us a third alternative. Instead of deducing \(P \rightarrow Q\) directly or using Contradiction, we can deduce the contrapositive \(\neg Q \rightarrow \neg P\). Using CP for this rather than the original conditional, we would suppose \(\neg Q\) and prove \(\neg P\).

On the surface of it, this also looks like a Proof by Contradiction is taking place. For aren’t we given \(P\) and asked to prove \(Q\)? If we then suppose its opposite \(\neg Q\) and end up with \(\neg P\), which contradicts \(P\), aren’t we concluding \(Q\) via NE?

No. We certainly could first suppose \(P\) and then suppose \(\neg Q\), but if we can prove \(\neg P\) from \(\neg Q\), then by CP we already have \(\neg Q \rightarrow \neg P\) and Conpsn gives us our conclusion, all without ever supposing \(P\) or using NE: a simpler proof overall. Thinking of Conpsn as Proof by Contradiction merely adds lines to the proof and possibly a layer of unnecessary subproof structure. In deriving \(\neg P\) from \(\neg Q\), we can’t be contradicting \(P\) if we never assumed \(P\) in the first place. \(P\) is not automatically given; we must suppose it in order to use it, as we do in proving \(P \rightarrow Q\) via CP.

The faulty identification of Contraposition as a form of Proof by Contradiction arises, it seems, when people fail to recognize the difference between temporary suppositions assumed for the sake of argument and the premises of an argument. \(P\) is not a premise in the deduction of \(P \rightarrow Q\); it is at most a temporary supposition.

Here is an example that exhibits the difference between the two types of argumentation.
EXAMPLE 1.8 - 7

Prove that if \( n^2 \) is even, then \( n \) is even.

**Solution**

We will prove this first using *Proof by Contradiction*, then using *Proof by Contraposition*.

**Proof #1:** (By *Contradiction*)

Suppose that \( n^2 \) is even, but that \( n \) is odd.

Then \( n = 2k + 1 \) for some integer \( k \).

Thus \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \), so \( n^2 \) is odd.

But this contradicts the fact that \( n^2 \) is even.

Therefore \( n \) is not odd; \( n \) is even. ■

**Proof #2:** (By *Contraposition*)

We show the contrapositive: if \( n \) is odd, then \( n^2 \) is odd.

So suppose \( n \) is odd.

Then \( n = 2k + 1 \) for some integer \( k \).

Thus \( n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 \), so \( n^2 \) is odd. ■

Note in this example that the second proof simplifies the first, merely by deleting certain more or less unnecessary steps. This kind of thing happens more often than you might suspect, even in mathematics textbooks. It may not always be possible to convert a *Proof by Contradiction* to one by *Contraposition*, but you should be alert to this possibility. Directness, elegance, and simplicity of style are factors that ought to be taken into consideration in devising a proof. *Proof by Contraposition* is generally superior to *Proof by Contradiction* in these respects.

*Proof by Contraposition* may seem like an indirect way to prove \( P \rightarrow Q \) (some classify it as an indirect proof), but it is far more directed than *Proof by Contradiction*, for in using *CP* to get \( \neg Q \rightarrow \neg P \), you assume \( \neg Q \) and attempt to deduce \( \neg P \). You can thus use the *Backward Method* to help construct the subproof. In the corresponding subproof for *Proof by Contradiction*, however, no *Backward Method* is available. You have no idea what the contradiction will turn out to be; you are not necessarily looking to derive \( \neg P \) in your subproof.

*Proof by Contraposition* is also more instructive than *Proof by Contradiction*, since it shows how negating the consequent forces the negation of the antecedent. *Proof by Contradiction* only tells you that supposing the negation of the desired conclusion while assuming the antecedent leads you into trouble.

**Ruminations on Indirect Proof Vs. Direct Proof**

In order to compare indirect and direct proofs more generally, we will first discuss when *Proof by Contradiction* is useful. It is clear from the two rules we have adopted (\( NI \) and \( NE \)) that *Proof by Contradiction* covers conclusions of all possible sentence forms. Whether or not a sentence is a negation, you can always attempt to prove it by showing the absurdity of its opposite. Nevertheless, it seems to be particularly useful when the result to be proved is itself a negation (\( NI \)). Some propositions in mathematics, such as the one asserting the irrationality of \( \sqrt{2} \), seem naturally to lend themselves to proofs by contradiction. A direct proof may be difficult to discover for such a proposition or, for all we know, may not even exist.

*Proof by Contradiction* is also useful when the opposite of the conclusion, assumed as a temporary supposition, can be readily combined with the given premises to derive other sentences. It often happens that the opposite of the conclusion does indeed provide fruitful
information to be used in conjunction with what is given. In fact, there will be times when you will not see any way to get going on a proof until you’ve assumed the opposite of the conclusion you want to show. Having Replacement Rules for simplifying various kinds of complex negations makes Proof by Contradiction a versatile method of proof.

Proof by Contradiction can be used as our overall strategy for deducing a proposition, but it is available for use at any time. In some instances, we may wait to use Proof by Contradiction until it looks promising or needed. We begin our deduction by using CP or some other rule of inference and then revert to NI or NE in the middle of the proof to establish the claim of some subproof instead of the final result.

From the point of view of overall proof strategy, Proof by Contradiction has both strengths and weaknesses. Being permitted to assume an additional sentence is a definite advantage. More premises are generally better than fewer. However, remember that what we are assuming is the exact opposite of what we want; it’s not like we’ve been granted an additional assumption that will lead forward to the given conclusion. And so, once we have denied the conclusion we really want, we have lost the beacon of the Backward Method of Proof Construction, and that is a heavy loss. Our task thus becomes one of deriving a contradiction, but without any knowledge of what or where the contradictory sentences will be. So while we have more sentences to work with than in a direct proof, we really have no definite idea where we are heading. Proof by Contradiction, therefore, is less directed than an ordinary direct proof. We may decide to use NI or NE on the basis of a Backward-Forward Proof Analysis of a certain proposition, but once we are embarked upon the real argumentation inside the subargument, we are moving off into an uncharted region with only the Forward Method at our disposal. What’s more, we will often not be able to use a realistic diagram to help us develop our proof, because what we are assuming is (presumably) false and so may be difficult or impossible to illustrate in a given situation.

There is another disadvantage to Proof by Contradiction as well. In assuming the opposite of the desired conclusion, we end up showing that the supposition is absurd. This gives us reason for believing the conclusion on negative grounds – “because otherwise this and its opposite would be the case, which is impossible.” Positive grounds for the conclusion, however, are still lacking. In some hard-to-define sense, an indirect proof contains less information than a direct one, which shows more directly how the conclusion is linked up with the premises. In this sense indirect proofs less clearly demonstrate their results. To put it a bit too simply, a Proof by Contradiction shows that something follows, but not why it follows. It is also for this reason that constructivist mathematicians shy away from or even take exception to certain proofs by contradiction.

Nevertheless, mathematicians are unwilling to give up Proof by Contradiction. G. H. Hardy, in his book A Mathematician’s Apology (1940), goes so far as to say that proof by reductio ad absurdum “is one of a mathematician’s finest weapons. It is a far finer gambit than any chess gambit: a chess player may offer the sacrifice of a pawn or even a piece, but a mathematician offers the game.” Twentieth century intuitionists, such as L. E. J. Brouwer, on the other hand, dispute the universal validity of such inferences because of their connection to the Law of Excluded Middle, which they reject as a general law. They allow Proof by Contradiction to be used only in certain situations. David Hilbert’s rejoinder to them was that “Forbidding a mathematician to make use of the principle of excluded middle [and hence its ally, Proof by Contradiction] is like forbidding an astronomer his telescope or a boxer the use of his fists” (quoted by H. Weyl in C. Reid’s Hilbert, 1970). Clearly, this is an issue in mathematics and logic that not everybody agrees on.
EXERCISE SET 1.8

Problems 1-5: Soundness of Inference Rules
Show that the following Int-Elim and Replacement Rules are sound.

1. **Double Negation** ($DN$): $\neg\neg P \vdash P$

2. **De Morgan’s Rules** ($DeM$)
   a. $\neg(P \land Q) \vdash \neg P \lor \neg Q$
   b. $\neg(P \lor Q) \vdash \neg P \land \neg Q$

3. **Negative Biconditional** ($Neg Bicndnl$): $\neg(P \leftrightarrow Q) \vdash (P \land \neg Q) \lor (\neg P \land Q)$

4. **Negative Conditional** ($Neg Cndnl$): $\neg(P \rightarrow Q) \vdash P \land \neg Q$

5. Demonstrate the soundness of $NI$ and $NE$ in the following ways.
   a. First show, given the derivation setup of $NI$ (the parts above the doubleunderline) and the rule $CP$, that you can conclude $P \rightarrow Q \land \neg Q$. Then show that $P \rightarrow Q \land \neg Q \vdash \neg P$ via an extended truth table.
   b. Similarly, show that the proof procedure of $NE$ is sound: consider its derivation setup, and then show that $\neg P \rightarrow Q \land \neg Q \vdash P$.

Problems 6-8: Completing Deductions
Fill in the reasons for the following deductions.

6. $P \rightarrow (Q \rightarrow R), \neg R \vdash \neg P \lor \neg Q$

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7. $P \lor Q \vdash (P \rightarrow Q) \rightarrow Q$

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<td>$(P \rightarrow Q) \rightarrow Q$</td>
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*7. $P \lor Q \vdash (P \rightarrow Q) \rightarrow Q$
### Problems 9-10: Logical Implication and Conclusive Deductions

Determine whether the following claims of logical implication are true or false. Then show where the deductions are inconclusive by pointing out where rules of inference are being used incorrectly.

9. \((P \to Q) \to [(P \to \neg Q) \to \neg P]\)

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<th>Premise</th>
<th>Reason</th>
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<td>((P \to \neg Q) \to \neg P)</td>
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<td>10</td>
<td>((P \to Q) \to [(P \to \neg Q) \to \neg P])</td>
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10. \((P \land Q) \to R, R \to S \models P \land \neg S \to \neg Q\)

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<th>Step</th>
<th>Premise</th>
<th>Reason</th>
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<td>Reit 1</td>
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<td>MP 8, 7</td>
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<td>(R \to S)</td>
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<td>14</td>
<td>(P \land \neg S \to \neg Q)</td>
<td>CP 4-13</td>
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### Problems 11-12: Exploration

Briefly explore and discuss the following topics.

11. Use an encyclopedia or a source on the history of mathematics or philosophy to look up Zeno’s paradoxes. Summarize one of these in your own words and explain how Proof by Contradiction enters into it.

12. Explain in your own words why Proof by Contradiction is called indirect proof. What is your reaction to the use of this method of proof in mathematics? Do you think indirect proofs show their results as clearly as direct proofs? Why or why not? Discuss the benefits and drawbacks to using indirect proof.
Problems 13-16: Negating Mathematical Propositions
Use Replacement Rules governing negated sentences to work the following problems.

13. An element is a member of the union $A \cup B$ of two sets $A$ and $B$ iff it is a member of $A$ or a member of $B$. What must be done to show an element is not a member of $A \cup B$? What Replacement Rule is involved in obtaining the necessary negation?

14. Two functions, $f$ and $g$, are inverses iff $g(f(x)) = x$ for all $x$ in the domain of $f$ and $f(g(y)) = y$ for all $y$ in the domain of $g$. What must be done in order to show that two functions $f$ and $g$ are not inverses of one another? What Replacement Rule is involved in obtaining the necessary negation?

15. Tell what must be shown in order to prove that the following result is false: if a function is continuous, then it is differentiable. Which Replacement Rule is involved in obtaining the necessary negation?

16. A series $\sum a_n$ is conditionally convergent iff it is convergent but not absolutely convergent. If you know that a series is not conditionally convergent, what might it be? What Replacement Rules justify your conclusion?

Problems 17-25: Deductions
Construct deductions, where possible, for the following problems, using only the Int-Elim Rules discussed up to this point. If a deduction cannot be constructed, explain why.

*17. $P \vdash \neg \neg P$

*18. $Q, \neg (P \land Q) \vdash \neg P$

*19. $P \land \neg P \vdash Q$

*20. $P \land \neg Q \vdash \neg (P \rightarrow Q)$

EC 21. $\neg (P \rightarrow Q) \vdash P \land \neg Q$

22. $\vdash (\neg P \rightarrow P) \vdash P$

23. $P \land Q \rightarrow R \vdash P \land \neg R \rightarrow \neg Q$

*24. $P \rightarrow (Q \land R), Q \rightarrow S, \neg S \vdash \neg P$

EC 25. $(P \rightarrow Q) \rightarrow P \vdash P$

Problems 26-32: More Deductions
Show that the following deduction claims hold, using any Inference Rules available to you so far.

26. $P \lor Q, \neg P \vdash Q$ \hspace{1cm} [Prove this using Cndnl.]

27. $\neg (P \land Q), Q \leftrightarrow R \vdash P \rightarrow \neg R$

28. $(P \land \neg Q) \rightarrow (\neg Q \land R), R \rightarrow \neg P \vdash P \rightarrow Q$

*29. $P \rightarrow R, Q \rightarrow \neg R \vdash P \rightarrow \neg Q$

30. $P \land Q \rightarrow R \land (P \rightarrow R) \lor (Q \rightarrow R)$

31. $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R$

32. $\neg Q \lor \neg R, P \rightarrow Q \land R \vdash \neg P$

33. $\vdash P \lor \neg P$ \hspace{1cm} [Prove this using LNC and DeM.]

Problems 34-38: Relations Among Inference Rules
Explore the connections between the following Inference Rules.

34. Redundancy of Contraposition
Show that Conpsn can be dropped from our Deduction System, provided we include MP and CP along with NI and NE; i.e., show that $P \rightarrow Q \vdash \neg Q \rightarrow \neg P$ using only these inference rules.

35. Proof by Contradiction and Double Negation
Show that the conclusion of NE can be derived from its set-up using the rules NI and the direction of DN that asserts $\neg \neg P \vdash P$. Thus, NE can be eliminated from our rules of inference without a loss of deductive power, provided we have NI and DN. Hence NE and one direction of DN are alike disputed by intuitionist mathematicians.
36. Law of Non-Contradiction, Law of Excluded Middle, and Double Negation
Show that \( \neg(P \land \neg P) \vdash P \lor \neg P \), given, among other things, the direction of Double Negation that asserts \( \neg \neg P \vdash P \). Without DN, what is the most that can be concluded from \( \neg(P \land \neg P) \)?

37. De Morgan’s Laws and Double Negation
Using the first rule of DeM and DN, show that the second rule also holds: \( \neg(P \lor Q) \vdash \neg P \land \neg Q \).

*38. Intuitionistic Scruples
According to intuitionistic scruples, \( \neg \neg P \not\implies P \). However, using the intuitionistically sound direction of DN (that of Exercise 17) or NI, show that \( \neg \neg \neg P \vdash \neg P \) as well as \( \neg P \vdash \neg \neg \neg P \). Thus, negative sentences are very different from positive sentences for intuitionists.

Problems 39-43: Deducing Everyday Arguments
Symbolically formulate the following arguments. Give a key for your argument, using the symbols indicated for the given positive sentences. If an argument is valid, derive the conclusion from the premises. Else, if it is invalid, find a truth value assignment that shows it to be invalid.

*39. If the police mount a Sting operation, they will be Encouraging crime. If the police do not mount a Sting operation, crime will Increase. Law and order are upheld iff crime is not Encouraged and crime does not Increase. Therefore Law and order cannot be upheld. \( [S, E, I, L] \)

40. If a Republican is elected, Taxes will be cut. If Taxes are cut, the Budget will not be balanced. But the Budget will not be balanced. Therefore a Republican will be elected. \( [R, T, B] \)

41. If a girl is a Minor and obtains a legal Abortion, she must have Parental consent. Therefore, if she legally obtains an Abortion without her Parents’ consent, she must not be a Minor. \( [M, A, P] \)

42. The weather is Cold if it is Winter time. If the weather is Cold but there has been insufficient Snow cover, then the Rhododendron bushes will not be protected. It is Winter time, and the Rhododendron bushes are protected. Therefore, there has been sufficient Snow. \( [W, C, S, R] \)

43. If additional road Taxes are placed on gasoline, the Price of fuel will rise, unless fuel production Costs decrease. But if Roads are to be improved, additional road Taxes must be placed on gasoline. And if the Price of fuel rises, truck Drivers will not be happy. So if production Costs do not decrease, either truck Drivers will be unhappy or the roads will not be improved. \( [T, P, C, R, D] \)

Problems 44-47: Irrationality Proofs
Work the following mathematical proofs using proof by contradiction.

44. Give another proof for the irrationality of \( \sqrt{2} \), as follows. Begin as in the text, but continue after line (*), which asserts that \( 2n^2 = m^2 \), by considering the possible number of factors of 2 contained in the prime factorizations of the right and left sides of the equation. Derive a contradiction from this and then conclude that \( \sqrt{2} \) must be irrational.

45. Prove that \( \sqrt{3} \) is irrational. Use an argument similar to one that proved the irrationality of \( \sqrt{2} \).

46. For which \( n \) is \( \sqrt{n} \) irrational? Why?

47. Prove that \( \sqrt[3]{2} \) is irrational.

Problems 48-53: The Pigeonhole Principle
Work the following mathematical proofs using proof by contradiction.

48. Using Proof by Contradiction in a paragraph-style proof, prove the so-called Pigeonhole Principle: if \( m \) objects (pigeons, letters, or whatever) are distributed to \( n \) containers (pigeonholes) and \( m > n \), then at least one pigeonhole has more than one object in it.

49. Apply the Pigeonhole Principle (Exercise 48) to show that there must be two New Yorkers with exactly the same number of hairs on their heads (though you would be hard pressed to locate them!). You may assume that there are more people living in New York than the total number of hairs possible on any one person’s head.
50. Show via the Pigeonhole Principle that given any set of five integral lattice points \((m, n)\) there is at least one integral lattice point (not necessarily one of the five) lying on one of the line segments connecting these points. [A point \((m, n)\) is an integral lattice point iff both \(m\) and \(n\) are integers.] Does this result hold if ‘five’ is replaced by ‘four?’ What would the analogous three-dimensional result be?

51. Prove that if \(\triangle ABC\) is a triangle having area 1 square unit and containing 9 distinct points, then 3 of these points must form a triangle whose area is less than or equal to 1/4 square unit. Hint: consider the triangles formed by using the mid-points of the sides as vertices and apply the Pigeonhole Principle.

52. Prove, using the Pigeonhole Principle, that in every set of 12 distinct two-digit numbers there are at least two numbers whose difference is a two-digit number having the same digits.

53. Show via the Pigeonhole Principle that given any set of \(n\) integers, some subset (possibly just a single number) of this set yields a sum that is divisible by \(n\). Hint: try \(n = 2, 3, 4\) to get started and then work the problem in general by considering what sorts of sums might arise relative to \(n\).

Problems 54-56: Mathematical Proofs by Contradiction

Work the following mathematical proofs using proof by contradiction.

54. Proof by Contradiction vs. Proof by Contraposition
   a. If \(2 \not| n\), then \(6 \not| n\). Put your argument into a formal proof diagram and identify the rules of inference you are using.
      Information needed: \(6 | n\) iff \(2 | n\) and \(3 | n\).
   b. Rework part a using Proof by Contraposition instead of Proof by Contradiction. Compare your proofs.
      Which one do you prefer? Why?

55. Show that the sum \(q + i\) of a rational number \(q\) and an irrational number \(i\) is irrational. Write your argument in paragraph style, but note explicitly how Proof by Contradiction enters into it.
   Hint: suppose that \(q + i\) is rational and derive a contradiction, knowing what you do about rational numbers.

56. Show that there are infinitely many prime numbers, using Proof by Contradiction. If you run stuck, locate and read a proof of this result in any textbook on number theory; or read it from Euclid’s Elements, Book IX, Proposition 20. Does your proof provide a definite procedure for generating an infinite list of prime numbers?

1.8-15
HINTS TO STARRED EXERCISES 1.8

7. [No hint.]
8. [No hint.]
10. To check whether the conclusion follows, see whether it is possible to make the conclusion F while all the premises are T. To evaluate the deduction, check how the suppositional arguments are being used.
16. First formulate the sentence symbolically. Then use \( NBE \) on the biconditional to get started and \( DeM \) to continue.
17. Use \( NI \).
19. Use \( NE \).
20. Use \( NI \).
24. Use \( NI \).
29. There are a couple of proof strategies that will work here, both direct and indirect.
38. To show \( \neg\neg\neg P \vdash \neg P \), use \( NI \).
39. The argument is valid; use \( NI \) to deduce the conclusion.
1.9 Int-Elim Rules for OR

In building up our Natural Deduction System for SL, we have adopted Int-Elim Rules related to the connectives “and”, “if-then”, “iff”, and “not”, and we have added to them a number of Replacement Rules based upon some important logical equivalents. The only rules our inference system still lacks are those governing the connective “or”. These will be incorporated in this section, after which our Deduction System will be complete.

As with the other connectives, we will adopt both basic Introduction and Elimination Rules, which we will discuss in the opposite order. And we will include a few Replacement Rules of a more algebraic nature, ones for expanding or contracting formulas involving “or”.

**Elimination Rule for OR: Disjunctive Syllogism**

We can draw a conclusion from a disjunction in two or three different ways. The simplest and the most widely used inference rule is traditionally called *Disjunctive Syllogism,* though a more descriptive name for it might be *Exclusion* or *Ruling Out Alternatives*. We will adhere to the standard terminology and abbreviate it by *DS*.

*Disjunctive Syllogism* proves one of a disjunction’s alternatives by ruling out the other one. It is used, therefore, whenever you know both that a disjunction is the case and also that one of the disjuncts is *not* the case. We include four forms under *DS* since the type of inference drawn is exactly the same in all of them.

Schematically, *DS* proceeds as follows:

\[
\begin{array}{c|c}
    DS & P \lor Q \\
    & \neg P \\
    & -P \\
    & Q \\
\end{array}
\]

\[
\begin{array}{c|c}
    DS & P \lor Q \\
    & \neg Q \\
    & -Q \\
    & P \\
\end{array}
\]

\[
\begin{array}{c|c}
    DS & -P \lor Q \\
    & P \\
    & \neg P \\
    & Q \\
\end{array}
\]

\[
\begin{array}{c|c}
    DS & P \lor -Q \\
    & Q \\
    & -P \\
    & P \\
\end{array}
\]

These rules are clearly sound (see Exercise 1). We can also show the soundness of *DS* by showing that any conclusion generated by *DS* can be deduced with other inference rules (see Exercises 9-10).

*Disjunctive Syllogism* is essentially *Modus Ponens* in the language of disjunction. This is easiest to see by looking at the third form of *DS*. If we were to replace the disjunction \( \neg P \lor Q \) with its conditional equivalent \( P \to Q \), we would have precisely *MP*. The connection between *DS* and *MP* is interesting to note from the point of view of the rule’s importance, for as you may recall, *MP* was earlier judged to be a central rule for working with conditional sentences.

The equivalence between *DS* and *MP* can be established even if we were to keep only the first two forms of *DS*. This is explored further in Example 1 and Exercise 8; here the intuitionistically valid direction of *DN* needs to be used to demonstrate the connection. *DS* is thus the counterpart to *MP* for intuitionists as well as for others.

**EXAMPLE 1.9 - 1**

Show that \( P \to Q \), \( P \vdash Q \) without using *MP*.

* This form of reasoning has little to do with the ordinary syllogisms of Aristotelian logic. It is called ‘disjunctive syllogism’ because it was originally appended to classical syllogistic logic as an additional form of reasoning.
Solution

The following proof diagram establishes the claim.

<table>
<thead>
<tr>
<th>No</th>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P → Q</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>P</td>
<td>Prem</td>
</tr>
<tr>
<td>3</td>
<td>¬P ∨ Q</td>
<td>Cndnl 1</td>
</tr>
<tr>
<td>4</td>
<td>¬¬P</td>
<td>DN 2</td>
</tr>
<tr>
<td>5</td>
<td>Q</td>
<td>DS 3, 4</td>
</tr>
</tbody>
</table>

The next example uses DS as its main proof strategy, but it first needs some preparation: in order to use DS we need to show that one of the disjuncts fails to hold. This is done via NI.

EXAMPLE 1.9-2

Show that P → Q ∧ R, P ∨ S, ¬Q ⊨ S.

Solution

The following proof diagram establishes the claim.

<table>
<thead>
<tr>
<th>No</th>
<th>Step</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P → Q ∧ R</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>P ∨ S</td>
<td>Prem</td>
</tr>
<tr>
<td>3</td>
<td>¬Q</td>
<td>Prem</td>
</tr>
<tr>
<td>4</td>
<td>P</td>
<td>Spsn for NI</td>
</tr>
<tr>
<td>5</td>
<td>P → Q ∧ R</td>
<td>Reit 1</td>
</tr>
<tr>
<td>6</td>
<td>Q ∧ R</td>
<td>MP 5, 4</td>
</tr>
<tr>
<td>7</td>
<td>Q</td>
<td>Simp 6</td>
</tr>
<tr>
<td>8</td>
<td>¬Q</td>
<td>Reit 3</td>
</tr>
<tr>
<td>9</td>
<td>¬P</td>
<td>NI 4-8</td>
</tr>
<tr>
<td>10</td>
<td>S</td>
<td>DS 2, 9</td>
</tr>
</tbody>
</table>

Disjunctive Syllogism is applied in a wide variety of different contexts. Inductive inferences in natural science are closely related to this rule, except that scientists generally don’t know that all alternatives have been uncovered. Thus, when they rule out several possibilities to leave the one they infer to be the case, their conclusion only has a certain degree of probability attached to it, not absolute certainty. It is DS, too, that justifies (insofar as is possible) most detective inferences made about who did what to whom and why (see Exercise 64). Sherlock Holmes held this rule in high regard: “Eliminate all other factors, and the one which remains must be the truth.” DS is also the main principle underlying much of what is called matrix logic on the elementary school level (see Exercise 63) and all Sudoku puzzle-playing decisions.

In mathematics DS is also widely used. To give just one example, in solving an equation f(x) = 0, possibly subject to some side conditions or constraints, you might first obtain all possible solutions c₁, . . . , cₙ to the equation and then rule out the ones that are extraneous or do not satisfy the constraints of the problem (see Exercise 73). This is an application of DS to a generalized disjunction.

Elimination Rule for OR: Proof by Cases

A second Elimination Rule is nearly as important as the one we just looked at, but it is a bit more complex. This rule is traditionally called Constructive Dilemma. Mathematicians are more familiar with this proof strategy under the equally descriptive name Proof by Cases, so we will use that and abbreviate it as Cases. An argument constructed according to this rule
derives a sentence $R$ on the basis of a disjunction $P \lor Q$ together with two separate subproofs of $R$ from $P$ and $Q$. *Proof by Cases* is thus another suppositional rule. In schematic form it proceeds as follows:

```
Cases

\[
P \lor Q
\]

\[
P
R
\]

\[
Q
R
\]

\[
R
\]
```

The traditional terminology for this rule (*Constructive Dilemma*) can be nicely explained in behavioral terms. Suppose you face a set of alternatives (the disjuncts), each of which entails the same, possibly unsavory, result (the conclusion). Then you’re faced with a real dilemma as to which action to take.

A classic example from ancient Greek philosophy illustrates this phenomenon. Euathlus ($E$) is trained as a lawyer by the Sophist Protagoras ($P$), with the mutual understanding that $E$ will pay for his training as soon as he wins his first case. When $E$ decides not to take up his profession, he is taken to court by $P$, who argues that if $E$ wins he must pay according to their agreement, while if $E$ loses he must pay according to the court’s ruling. Thus, in any case, $E$ must pay up. $E$ faces a real dilemma, which it seems he cannot escape, given the agreement made with $P$.

The reason why the alternate term *Proof by Cases* is used for this rule of inference should be fairly obvious: you prove $R$ by supposing in turn each of two alternatives, $P$ and $Q$, showing that in each case $R$ follows. Since one of these two cases certainly holds, having already asserted their disjunction, you are permitted to conclude $R$.

*Proof by Cases* is a sound rule of inference. If you are able to show that the only two cases possible both entail the same result, then that result must be true, even though you may not know which of the two cases holds. The soundness of such a rule can be demonstrated by showing that *Cases* is dispensable; that is, by showing that any argument established by *Cases* can also be deduced by means of the rules of inference for SL already accepted (see Exercise 56). The soundness of *Cases* can also be demonstrated more directly, assuming that $CP$ is sound, by showing that $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R$ (see Exercise 7).

We will illustrate how this rule of inference is used by giving several examples, starting with two from SL proper.

EXAMPLE 1.9-3

Show that $P \lor Q, P \rightarrow R, Q \rightarrow R \vdash P \land R$.

**Solution**

The following proof diagram establishes the claim. Note that the second subproof has been slightly abbreviated by citing the first subproof. This is legitimate; nothing appears as a premise of the first subproof that is not available in the second where it is cited.

* Of course, $E$ might still try to wriggle out of his responsibility by hiring a lawyer to defend him instead of taking up his own case, but barring a win in this way, $E$’s agreement forces him to pay up. However, see also Exercise 55 for a creative counterattack that can be taken by $E$. 

1.9-3
1. $P \lor Q$  \hspace{1cm} Prem
2. $P \rightarrow R$  \hspace{1cm} Prem
3. $Q \rightarrow P$  \hspace{1cm} Prem

4. $P$  \hspace{1cm} Spsn 1 for Cases
5. $P \rightarrow R$  \hspace{1cm} Reit 2
6. $R$  \hspace{1cm} MP 5, 4
7. $P \land R$  \hspace{1cm} Conj 4, 6

8. $Q$  \hspace{1cm} Spsn 2 for Cases
9. $Q \rightarrow P$  \hspace{1cm} Reit 3
10. $P$  \hspace{1cm} MP 9, 8
11. $P \land R$  \hspace{1cm} Subproof #1 10
12. $P \land R$  \hspace{1cm} Cases 1, 4-7, 8-11

Using *Cases*, we will now prove one direction of the $\lor$-form of *Cndnl*. The other direction will be worked once we have rules for introducing disjunctions (see Example 8).

**EXAMPLE 1.9-4**

Show that $\neg P \lor Q \vdash P \rightarrow Q$ without using *DS* or *Cndnl*, using *Cases* instead.

**Solution**

The following deduction establishes the claim. Note that in the initial subproof we aim at first proving the contrapositive, since that fits better with the negative case under consideration, while in the other subproof we prove the conditional as it is given.

1. $\neg P \lor Q$  \hspace{1cm} Prem
2. $\neg P$  \hspace{1cm} Spsn 1 for Cases
3. $\neg Q$  \hspace{1cm} Spsn for CP
4. $\neg P$  \hspace{1cm} Reit 2
5. $\neg Q \rightarrow \neg P$  \hspace{1cm} CP 3-4
6. $P \rightarrow Q$  \hspace{1cm} Conpsn 5

7. $Q$  \hspace{1cm} Spsn 2 for Cases
8. $P$  \hspace{1cm} Spsn for CP
9. $Q$  \hspace{1cm} Reit 7
10. $P \rightarrow Q$  \hspace{1cm} CP 8-9
11. $P \rightarrow Q$  \hspace{1cm} Cases 1, 2-6, 7-10

In formal proofs, such as the ones just given, each case is normally argued separately, the final conclusion resting upon the validity of both jointly. In an informal proof by cases, though, the argument may proceed in a less obvious way. You may only be alerted to the fact that there are several cases involved by reading phrases like ‘on the one hand’ and ‘on the other hand’. Informal proofs by cases are often less complete than formal proofs as well. You may only find one case proved in detail, particularly if the second case proceeds similarly, with only a few minor modifications. In this context you may see the phrase ‘without loss of generality [or its abbreviation *wolog*], suppose such and such is the case’. The understanding implicit
among mathematicians when such a phrase is encountered is that the other case is entirely analogous, so why belabor the proof by doing the same deduction twice? Or, the second case may be dismissed with a phrase such as ‘The other case is proved similarly’ or ‘Apply Case 1 to such and such to see that the conclusion follows’. Example 3 above does something like this by citing the first subproof. When you come across statements like this, it is an invitation to you to take pencil and paper in hand to work out the second subproof for yourself. Occasionally you will find a non-trivial wrinkle in the deduction that costs you a bit of time and effort and makes you wonder why the writer thought the subproofs were so similar and whether he or she even worked it out. Well, that keeps life interesting and mathematicians honest.

Proof by Cases is a favorite and valuable proof strategy in many different fields of mathematics. We can distinguish essentially two different kinds of proofs that use cases in mathematics. One of them (the first) is a genuine application of Proof by Cases, while the other one is a bit different. We will discuss both types, but spend more time on the former.

The first type of cases proof is based upon mutually exclusive alternatives. This occurs, for example, in connection with propositions that mention an object, property, or relation that has been defined in a piecemeal fashion; that is, in terms of different cases. This happens often enough: the absolute value function and various other functions are defined as split functions; the relation of “less than or equal to” is defined by means of two different alternatives; and so on. The way in which the particular definition is set up will determine what cases you need to consider in the proof of your proposition. We will illustrate this with a simple fact from the arithmetic of real numbers.

✠ EXAMPLE 1.9 - 5
Prove that $|x| \geq 0$ for all real numbers.

Solution

The absolute value function is defined by $|x| = \begin{cases} x : & x \geq 0 \\ -x : & x < 0 \end{cases}$.

To prove the given result, we will take cases based upon the definition.

For $x \geq 0$, $|x| = x$, so in this case the result holds by substitution: $|x| \geq 0$.

On the other hand, if $x < 0$, $|x| = -x$.

Since $x < 0$, $-x > 0$.

Thus, $|x| > 0$, so certainly $|x| \geq 0$. (Here we really need an ‘or’ Introduction Rule – the Addition Rule – that we have yet to discuss; see below.)

Hence in all cases $|x| \geq 0$.

Proof by Cases can be attempted whenever a disjunction occurs as a line in your proof, but in fact, as the last example illustrates, it is a prime candidate for consideration whenever the disjuncts are exclusive alternatives. This sort of Proof by Cases arises when the universe of discourse for a proposition can be partitioned into mutually exclusive subsets, such as evens and odds; primes and composites; rationals and irrationals; or negatives, zero, and positives. If being a member of each subclass entails the desired result, then the conclusion holds for all the objects in the class by Cases. Such a tactic is quite common in mathematics. In constructing an informal proof of a conclusion $R$, you may not explicitly state the disjunction that defines the possibilities, but it is that sentence in conjunction with the subsequent subproofs that allows you to conclude via Cases that $R$ holds without exception.

✠ EXAMPLE 1.9 - 6
Sketch a general proof procedure for the proposition the square of any non-zero real number is positive.
Solution

Since the non-zero real numbers can be decomposed into the set of negative numbers and the set of positive numbers, we would attempt to show the proposition is true for each of these cases.

By Cases, we would then conclude that the proposition is true in general for all non-zero real numbers.

Now, if you logically analyze the disjunctions used in such situations, you will notice that they can be thought of as instances of the Law of Excluded Middle: “x is a positive number or it is not a positive number”; “x is even or x is not even”, etc. The classification used might make it seem like both terms are affirmative ones (positive/negative; even/odd; prime/composite; etc.), but in reality, an object belongs to one of the subclasses iff it does not belong to the other: n is odd iff n is not even, and so on. The classification is exhaustive: an object must belong to one of the designated subclasses, and it belongs to one iff it does not belong to the other. Thus the disjunction that grounds our conclusion is really of the form P ∨ ¬P: n is positive or n is not positive, etc. Thus, a Cases argument here proves a sentence R on the basis of P ∨ ¬P, whether or not you have expressly stated it. Since R follows from P and also from ¬P, it must follow from P ∨ ¬P, according to Cases. Since this sentence is a tautology, it is not an additional premise, but it is nevertheless required for the argument to be complete.

The proof schema for such an argument would look as follows:

\[
\begin{array}{cccc}
i & P \lor \neg P & \text{LEM} \\
j & P & \text{Spsn 1 for Cases} \\
m & \cdots \\
n & \neg P & \text{Spsn 2 for Cases} \\
s & \cdots \\
t & R & \text{Cases i, j-m, n-s}
\end{array}
\]

In this sort of proof, there does not seem to be a uniform way by which to avoid the Law of Excluded Middle. It may be possible to rewrite such a proof so that it is not used, but as it stands, LEM appears in an essential way, generating the cases to be investigated. Indeed, as we have noted, there seem to be many situations in which such a tautology is both desirable and a rather natural way to make a derivation (see Exercise 18 and others). This dichotomizing divide-and-conquer approach often breaks a proof up into manageable pieces by focusing your attention on distinct subclasses of the universe of discourse. Using LEM often demands some creativity and insight on your part (how should we best split things up?), but it is a standard maneuver in the mathematician’s set of moves.

We have excluded tautologies from our Natural Deduction System for SL up to this point because they are generally not used in mathematical proofs, but having found a rather natural way in which they do occur, we will now make an exception. An instance of the Law of Excluded Middle can be asserted at any line in a proof without any prior premises, citing LEM as the reason. As we noticed in Section 1.8 (see Exercise 1.8-33), such a sentence can always be proved on the basis of no premises using the rules already present in our Deduction System, so we aren’t increasing the power of our Deduction System by adopting this rule, only its efficiency. The rule schema for LEM is the following:

\[
\begin{array}{cc}
\text{LEM} & P \lor \neg P \\
\end{array}
\]

1.9-6
There is a second form of proof that sometimes goes under the name *Proof by Cases*, but its logical basis and character are a bit different from what we have just discussed. Cases are considered, as before, but it is questionable whether such proofs are really based on a disjunction, because the alternatives involved are not mutually exclusive. In fact, they are usually hierarchically related, becoming ever more inclusive alternatives (prime numbers and positive integers; integers, rationals, and real numbers). Proofs that argue with such cases are peculiar and interesting from a logical standpoint because the final case all by itself suffices to prove the entire proposition, not just one part of it. Why, then, are the earlier cases even considered? The answer is that the final case needs the results of the earlier cases for its proof.

For instance, to prove a result such as \( a^m \cdot a^n = a^{m+n} \) for rational exponents, you first prove it for positive integers \( m \) and \( n \), and then you go on to prove it first for integers in general and then for rational numbers (see Section 3.1). In the latter proofs, you appeal to the fact that the law already holds for the case of positive integers. The proof for integers turns out to be a genuine *Proof by Cases* application, because negative numbers are treated as a separate alternative to positive integers. The proof for rational numbers, however, makes no attempt to treat fractional exponents separately from integer exponents. It works whether or not the exponent is fractional, using the fact that the law already holds for integers. This argues the same formal conclusion for different cases, but the full result isn’t really validated by *Proof by Cases*; it merely uses a proposition proved for a more limited class of objects to establish the same result in a broader context. While such proofs are not strictly in the *Proof by Cases* mold, they do use an important proof technique involving cases that all students of mathematics should be familiar with. Mathematicians sagely call this proof strategy ‘bootstrapping’ because you gradually pull yourself up by your bootstraps, as it were, proving the general case from the special.

**Introduction Rule for OR: Addition**

The simplest introduction rule for “or” is one that at first glance seems rather useless. It is the *Addition Rule* (*Add*), which permits you to conclude \( P \lor Q \) from knowing either \( P \) or \( Q \). Schematically, it is represented as follows:

\[
\begin{align*}
\text{Add} & \quad \frac{P}{P \lor Q} \quad \frac{Q}{P \lor Q}
\end{align*}
\]

Now why, you might wonder, would anyone want to conclude \( P \lor Q \) if they already know something stronger, namely, one of the disjuncts? Well, there actually are times in which this is desirable. We just saw one in Example 5. The result we wanted there was a disjunction (\(|x| \geq 0\); i.e., \(|x| > 0 \lor |x| = 0\)), while the conclusion we actually obtained in the case where \( x \) was negative was only one of the disjuncts: \(|x| > 0\). To obtain the desired generalization, we had to go on from there to assert the full disjunction, \(|x| \geq 0\). This requires *Addition*, as we noted. This step is usually omitted in an informal proof as being too obvious to state, but if all the steps of the argument are made explicit, *Add* must be cited as a justification for the inference.

The following formal argument establishing the deductive commutativity of \( \lor \) illustrates the use of *Add* in a *Proof by Cases*.

**EXAMPLE 1.9-7**

Show that \( P \lor Q \vdash Q \lor P \).
Solution

Note first of all that only one direction really needs to be proved here, for if we establish that \( P \lor Q \vdash Q \lor P \), it follows by substitution that \( Q \lor P \vdash P \lor Q \), too: simply interchange the formulas \( P \) and \( Q \) in the result gotten.

The proof of \( Q \lor P \) from \( P \lor Q \) goes as follows.

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<tbody>
<tr>
<td>1</td>
<td>( P \lor Q )</td>
<td>Prem</td>
</tr>
<tr>
<td>2</td>
<td>( P )</td>
<td>Spsn 1 for Cases</td>
</tr>
<tr>
<td>3</td>
<td>( Q \lor P )</td>
<td>Add 2</td>
</tr>
<tr>
<td>4</td>
<td>( Q )</td>
<td>Spsn 2 for Cases</td>
</tr>
<tr>
<td>5</td>
<td>( Q \lor P )</td>
<td>Add 4</td>
</tr>
<tr>
<td>6</td>
<td>( Q \lor P )</td>
<td>Cases 1, 2-3, 4-5</td>
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Introduction Rule for OR: Either-Or

A second Introduction Rule is known as Either-Or (EO). Whereas Addition is used when you know for sure that one of \( P \) or \( Q \) is the case and you want the disjunction \( P \lor Q \) instead, Either-Or is the rule to use when you want to prove a disjunction but are not sure which disjunct might be the case. Since this is the more usual situation, EO is used more frequently in mathematical proofs than Add. We will discuss this rule in some detail, since students are sometimes perplexed about its soundness when they first encounter it in deductions.

Either-Or proceeds in the following way. Since \( P \lor Q \) is the case whenever either of the disjuncts is true, \( P \lor Q \) holds even if one of them is not the case. In fact, if you can show that one of the disjuncts must be the case whenever the other one isn’t, then the full disjunction must follow. Schematically we have the following:

\[
\text{EO} \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 
\end{align*}

Here’s the rationale for this procedure. Certainly the subproof in the left hand scheme proves \( \neg P \rightarrow \neg Q \) (via CP); and since \( \neg P \rightarrow Q \rightarrow P \lor Q \) (use Cndnl and DN), it also establishes \( P \lor Q \). Analogously, \( \neg Q \rightarrow \neg P \rightarrow P \lor Q \), so the other inference is sound, too.

For those who still feel uneasy about the legitimacy of this inference scheme (don’t we need both subproofs to show \( P \lor Q \)?), we can argue the validity of the conclusion from the given set-up (for the left hand scheme, for instance) using LEM, Add, and Cases. For, according to LEM, the only two options available to us are \( P \) and \( \neg P \). If \( P \) is the case, then the conclusion \( P \lor Q \) follows by Add. On the other hand, if \( \neg P \) is the case, then since we know that \( \neg P \rightarrow Q \) by the set-up we are given, we can conclude \( Q \). And so again \( P \lor Q \) follows via Add. Since we have considered all possible cases, \( P \lor Q \) must follow (Cases).
Schematically, we have been arguing as follows:

\[
\begin{align*}
\text{i} & : P \lor \neg P & \text{LEM} \\
\text{j} & : P & \text{Spsn 1 for Cases} \\
\text{k} & : P \lor Q & \text{Add } j \\
\text{l} & : \neg P & \text{Spsn 2 for Cases} \\
\vdots \\
\text{p} & : Q & \text{Add } p \\
\text{q} & : P \lor Q & \text{Cases i, j-k, l-q}
\end{align*}
\]

Such an argument form could be used to replace *Either-Or*. What *EO* does, however, is to conclude the disjunction directly without such an elaborate deduction. If you can prove one disjunct from the negation of the other, infer the disjunction. Note, moreover, that you do this on the basis of just one such argument. It is not necessary to take each disjunct in turn and show that the other one follows from its negation; that would prove the result twice. Which disjunct you choose to negate depends largely upon what else you are given to work with. The negation of one of the disjuncts may deductively combine much more easily with the premises to give you the other disjunct than the other way around. For constructing formal proofs of this type, then, the *Forward Method of Proof Analysis* may be able to assist you in choosing a good strategy. If you fail to see which negation to begin with, try them both in turn and see which one generates what you need.

To illustrate *Either-Or* as well as the earlier rules for disjunction, we will finish with two examples from SL proper and one from mathematics. The second example again illustrates the use of a previously proved result.

**EXAMPLE 1.9 - 8**

Show that \( P \rightarrow Q, \neg P \rightarrow R \vdash Q \lor R \).

*Solution*

The following proof diagram establishes the claim.

\[
\begin{align*}
1 & : P \rightarrow Q & \text{Prem} \\
2 & : \neg P \rightarrow R & \text{Prem} \\
3 & : \neg Q & \text{Spsn for EO} \\
4 & : P \rightarrow Q & \text{Reit } 1 \\
5 & : \neg P & \text{MT 4, 3} \\
6 & : \neg P \rightarrow R & \text{Reit } 2 \\
7 & : R & \text{MP 6, 5} \\
8 & : Q \lor R & \text{EO 3-7}
\end{align*}
\]

**EXAMPLE 1.9 - 9**

Show that \( \neg(P \land Q) \vdash \neg P \lor \neg Q \) without using *DeM*.

*Solution*

The following proof diagram establishes the claim. To keep the argument brief, we will cite an example proved earlier.

1.9-9
1.9 - 10

Prove that \( ab = 0 \leftrightarrow a = 0 \lor b = 0 \).

\textbf{Solution}

Since the main connective here is \( \leftrightarrow \), we will use BI. In working the two subproofs, we will use rules for introducing and eliminating disjunctions. Try to identify them yourself (see Exercise 65).

We will assume as already known the proposition \( x \cdot 0 = 0 = 0 \cdot x \) for any real number \( x \).

\textbf{Proof}:

\( \rightarrow \): First suppose that \( ab = 0 \).
- If \( a \neq 0 \), then multiplying \( ab = 0 \) by \( 1/a \), which is well-defined, yields \( b = (1/a) \cdot 0 = 0 \).

\( \leftarrow \): Now suppose that \( a = 0 \lor b = 0 \).
- If \( a = 0 \), then \( ab = 0 \cdot b = 0 \).
- If \( b = 0 \), a similar result follows.

Thus \( ab = 0 \).

Therefore \( ab = 0 \leftrightarrow a = 0 \lor b = 0 \). \( \blacksquare \)

\textbf{Replacement Rules Involving Disjunctions}

In Section 1.5 we introduced three Replacement Rules for conjunction: \textit{Commutation}, \textit{Association}, and \textit{Idempotence}. These same rules hold for disjunction. They are used in informal arguments without thinking about it, but they should be cited when used in a formal deduction. They are schematized as follows.

\begin{align*}
\text{Comm} (\lor) & \quad P \lor Q :: Q \lor P \\
\text{Assoc} (\lor) & \quad P \lor (Q \lor R) :: (P \lor Q) \lor R \\
\text{Idem} (\lor) & \quad P \lor P :: P
\end{align*}

In addition to these rules, there are two Replacement Rules that govern how conjunctions and disjunctions can be distributed in complex sentences. The \textit{Distribution Rules} \((\text{Dist})\) are probably best thought of as giving algebra-like ways in which to expand or contract expressions involving \( \land \) and \( \lor \). These inferences must be consciously drawn in an argument in which they occur; they are not automatically applied by someone who hasn’t thought about them or isn’t familiar with the interaction of \( \land \) and \( \lor \). Even though they lack a degree of naturality, we will include them among our Replacement Rules because of their importance for simplifying and inter-relating compound sentences. There are four forms here. They are schematized and labeled as follows:

\begin{align*}
\text{Dist} (\land \text{ over } \lor) & \quad P \land (Q \lor R) :: (P \land Q) \lor (P \land R) \\
\text{Dist} (\land \text{ over } \lor) & \quad (P \lor Q) \land R :: (P \lor R) \lor (Q \land R) \\
\text{Dist} (\lor \text{ over } \land) & \quad P \lor (Q \land R) :: (P \lor Q) \land (P \lor R) \\
\text{Dist} (\lor \text{ over } \land) & \quad (P \land Q) \lor R :: (P \lor R) \land (Q \lor R)
\end{align*}
**EXERCISE SET 1.9**

**Problems 1 - 7: Soundness of Inference Rules**
Show that the following Int-Elim and Replacement Rules are sound.

1. **Disjunctive Syllogism (DS)**
   a. \( P \lor Q, \neg P \vdash Q \)
   b. \( P \lor Q, \neg Q \vdash P \)
   c. \( \neg P \lor Q, P \vdash Q \)
   d. \( P \lor \neg Q, Q \vdash P \)

2. **Law of Excluded Middle (LEM):** \( \vdash P \lor \neg P \)

3. **Addition (Add)**
   a. \( P \vdash P \lor Q \)
   b. \( Q \vdash P \lor Q \)

4. **Commutativity (Comm):** \( P \lor Q \vdash Q \lor P \)

5. **Associativity (Assoc):** \( P \lor (Q \lor R) \vdash (P \lor Q) \lor R \)

6. **Idempotence (Idem):** \( P \lor P \vdash P \)

7. **Proof by Cases (Cases)**
   a. Show that \( P \lor Q, P \rightarrow R, Q \rightarrow R \vdash R \).
   b. Explain how part a, together with the soundness of CP, show that Cases is sound.

8. Show that the final two forms for DS can be omitted, but that then the inference needs to be mediated by DN; i.e., prove the following, using only the first two forms for DS along with DN.
   a. \( \neg P \lor Q, P \vdash Q \)
   b. \( P \lor \neg Q, Q \vdash P \)

9. Show that \( P \lor Q, \neg P \vdash Q \) without using DS. This result, along with Example 1, establishes the equivalence of DS and MP.

10. Show that the following derivations hold, using only the first form of DS in combination with other rules of inference.
    a. \( P \lor Q, \neg Q \vdash P \)
    b. \( \neg P \lor Q, P \vdash Q \)
    c. \( P \lor \neg Q, Q \vdash P \)

**Problems 11 - 13: Completing Deductions**
Fill in the reasons for the following deductions.

11. \( P \rightarrow (Q \rightarrow R), \neg R \vdash \neg P \lor \neg Q \)

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*12. \((P \rightarrow Q) \rightarrow Q \vdash P \vee Q\)

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<td>4</td>
<td>\neg(P \rightarrow Q)</td>
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<td>(P \land \neg Q)</td>
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<td>6</td>
<td>(P)</td>
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<td>7</td>
<td>(P \vee Q)</td>
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*13. \(P \vee Q \vdash (P \rightarrow Q) \rightarrow Q\)

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<td>(P \rightarrow Q)</td>
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<td>(Q)</td>
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<td>10</td>
<td>((P \rightarrow Q) \rightarrow Q)</td>
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Problems 14-17: Contraction Rules
Show that the following results hold and explain in words what they mean about deductions that involve a contraction/absorption (proceeding left to right).

14. \(P \land (P \lor Q) \vdash P\)

15. \(P \lor (P \land Q) \vdash P\)

16. \(P \land (Q \lor \neg Q) \vdash P\)

17. \(P \lor (Q \land \neg Q) \vdash P\)

Problems 18-20: Logical Implication and Conclusive Deductions
Determine whether the following claims of logical implication are true or false. Then determine whether the deductions that are given are conclusive or inconclusive. Carefully point out each place where a rule of inference is being used incorrectly.

*18. \(P \lor Q \rightarrow \neg R, Q \lor R \vdash \neg R\)

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<td>Add 3</td>
</tr>
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<td>5</td>
<td>(P \lor Q \rightarrow \neg R)</td>
<td>Reit 1</td>
</tr>
<tr>
<td>6</td>
<td>\neg R</td>
<td>MP 5, 4</td>
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<td>7</td>
<td>(Q \lor R)</td>
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<td>11</td>
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<td>DS 10, 2</td>
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1.9-12
*19. \( P \rightarrow Q, R \rightarrow \neg Q, P \lor \neg R \vdash Q \)
   1 \( P \rightarrow Q \) Prem
   2 \( R \rightarrow \neg Q \) Prem
   3 \( P \lor \neg R \) Prem
   4 \( P \) Spsn 1 for Cases
   5 \( P \rightarrow Q \) Reit 1
   6 \( Q \) MP 5, 4
   7 \( \neg P \) Spsn 2 for Cases
   8 \( P \lor \neg R \) Reit 3
   9 \( \neg R \) DS 8, 7
   10 \( R \rightarrow \neg Q \) Reit 2
   11 \( Q \) MT 10, 9
   12 \( Q \) Cases 3; 4-6, 7-11

20. \( \neg P \land Q \rightarrow R, R \rightarrow P \vdash P \lor Q \)
   1 \( \neg P \land Q \rightarrow R \) Prem
   2 \( R \rightarrow P \) Prem
   3 \( \neg P \) Spsn for EO
   4 \( \neg P \land Q \) Add 3
   5 \( \neg P \land Q \rightarrow R \) Reit 1
   6 \( R \) MP 5, 4
   7 \( R \rightarrow P \) Reit 2
   8 \( P \) MP 7, 6
   9 \( P \lor Q \) Add 8
   10 \( P \lor Q \) EO 3-9

Problems 21-24: Simplifying Expressions Involving Negations, Conjunctions, and Disjunctions

Using the relevant Replacement Rules as well as any results from Exercises 14 - 17, show that the following hold.

21. \( P \lor (P \land \neg Q) \vdash P \)
22. \( P \land (\neg P \lor Q) \vdash P \land Q \)
23. \( (P \lor \neg Q) \land (\neg P \lor \neg Q) \vdash \neg Q \)
24. \( \neg(P \lor Q) \lor (P \land Q) \vdash (\neg P \lor Q) \land (\neg Q \lor P) \)

Problems 25-47: Deductions

Show that the following deduction claims hold, using any rules of inference available to you, unless otherwise instructed.

25. \( P \land Q \vdash P \lor Q \)
26. \( P \lor Q, P \rightarrow Q \vdash Q \) [Prove this directly, without using NE.]
27. \( P \lor Q, P \rightarrow R, R \rightarrow \neg P \vdash Q \)
28. \( P \lor Q, \neg(P \land R), R \vdash \neg Q \)
29. \( \neg P \rightarrow Q \vdash P \lor Q \) [Prove this without using Cndnl.]
30. \( P \lor Q, Q \rightarrow R \vdash P \lor R \)
31. \( \neg P \lor R, \neg Q \lor \neg R \vdash \neg P \lor \neg Q \)
32. \( P \rightarrow R, Q \rightarrow R \vdash P \lor Q \rightarrow R \)
*33. \( Q \rightarrow R \vdash P \lor Q \rightarrow P \lor R \)
34. \( \neg P \rightarrow P \vdash P \)
35. \( P \rightarrow ((Q \rightarrow R) \lor (\neg Q \rightarrow S)) \vdash P \rightarrow R \lor S \)
36. \( Q \rightarrow \neg R, \neg P \lor Q \land R \vdash \neg P \)
37. \( P \rightarrow Q \lor R, Q \rightarrow S, R \rightarrow S \vdash P \rightarrow S \)
*38. \( \vdash (P \rightarrow Q) \lor (Q \rightarrow P) \)
39. \( \vdash (P \rightarrow Q) \lor \neg Q \)
40. \( P \lor Q, P \lor \neg Q \vdash \neg P \)
41. \( P \land Q \rightarrow R, P \rightarrow R \lor Q \vdash R \lor S \)
42. \( \vdash (P \rightarrow Q) \lor (P \lor Q) \)
43. \( \neg P \rightarrow Q \lor R \lor S, P \rightarrow T, \neg R, S \rightarrow T \vdash Q \lor T \)
*45. \( P \rightarrow R, Q \rightarrow \neg R, \neg P \lor \neg Q \vdash \neg P \)
46. \( P \rightarrow Q \land R, Q \rightarrow \neg R \vdash \neg P \)
EC 47. \( P \leftrightarrow (Q \leftrightarrow P) \vdash \neg Q \)

Problems 48 - 54: Interderivability
Show that the following distribution-like interderivability claims hold, using any rules of inference available to you, unless otherwise instructed.
48. \( (P \rightarrow Q) \lor R \vdash (P \lor R) \rightarrow (Q \lor R) \)
49. \( (P \rightarrow Q) \lor R \vdash P \rightarrow (Q \lor R) \)
50. \( P \rightarrow (Q \lor R) \vdash (P \rightarrow Q) \lor (P \rightarrow R) \)
51. \( (P \rightarrow Q) \lor R \vdash (P \rightarrow Q) \lor Q \)
52. \( P \rightarrow (Q \lor R) \vdash (P \land \neg Q) \rightarrow R \)
*53. \( P \land Q \rightarrow R \vdash (P \lor R) \lor (Q \rightarrow R) \)
54. \( P \lor Q \rightarrow R \vdash (P \rightarrow R) \land (Q \rightarrow R) \)

*55. Develop the classic example of Euathlus vs. Protagoras (given in the text prior to Example 3), only this time so that the dilemma turns out favorable to Euathlus. Compare your argument with what is in the text and comment on the result.

56. Cases and Other Inference Rules
Show that Cases can be dropped from our Deduction System without any loss of deductive power; i.e., without using Cases, show \( P \land Q \lor \neg Q \rightarrow R \lor S \)
57. Either-Or and Law of Excluded Middle
Show how the rule Either-Or can be used to deduce LEM. Hence, EO is unacceptable to intuitionist logicians.
58. Show that the consequences generated by NE from its given set-up can also be obtained by NI in combination with LEM and DS.

*59. Proof by Contradiction, Addition, and Disjunctive Syllogism
Using the rules Add and DS, show that \( Q \land \neg Q \rightarrow R \) for any R.

Problems 60 - 62: Explorations with ‘Exclusive Or’
Work the following problems related to \( \lor \).
60. Which of the Int-Elim Rules for \( \lor \) are sound if \( \lor \) is replaced by \( \land \)? Give reasons in support of your answer.
61. Show that the following inference rule is sound: \( P \lor Q, P \lhd \neg Q \). Would this rule be sound if \( \lor \) were substituted for \( \land \)?

1.9-14
62. Show that the following conversion rule holds: \( P \lor Q, P \rightarrow R, Q \rightarrow S, R \lor S \models (R \rightarrow P) \lor (S \rightarrow Q) \). Would this rule be sound if \( \lor \) were substituted for \( \land \)?

**Problems 63-64: Disjunctive Syllogism Problems**

*Work the following problems involving Disjunctive Syllogism.*

63. **Matrix Logic Puzzle**

From the information given below, determine the name and class of the mathematics major taking logic. You do not have to put your reasoning into a formal proof, but explain how you arrived at your result. It may help you to keep track of your information if you use a chart labeled with the students’ names along the side and with their sex, class, and major along the top.

**Data:**

1. A logic class contains five students: Amy, Bill, Caitlin, Dave, and Evelyn. Two of these are boys, and three are girls; two are sophomores, two are juniors, and one is a senior; three are prospective mathematics teachers, one is a general mathematics major, and one is a computer science major.
2. At most one boy is a junior.
3. Two of the prospective mathematics teachers are juniors.
4. Caitlin is not the least interested in teaching.
5. The mathematics major is taking logic sooner than either Bill or Caitlin did.
6. Dave and Evelyn have different majors.
7. Amy and Bill belong to different classes.

*64. **Detective Reasoning***  

Security has been breached at American Federal Bank and a large sum of money has been taken. Police have assembled the following clues. Help them track down all the culprits by means of a formal deduction, using the letters indicated for the positive atomic sentences.

1. The suspects are a Teller, a Computer programmer, a Loan officer, and a Vice-President.
2. If it was the Computer programmer or the Teller, then one of the others was also involved.
3. If it was the Teller or the Vice-President, then it did not occur on the Weekend.
4. If Security was bypassed, then it was either the Computer programmer or the Vice-President.
5. If the break-in was Discovered later, then Security was bypassed.
6. The break-in occurred on the Weekend, but it was not Discovered until later.

**Problems 65-71: Mathematical Proofs**

*Work the following mathematical proofs, as indicated.*

*65. Analyze the logical structure of the proof given in Example 10, pointing out which Int-Elim Rules for \( \lor \) are used and where.*

*66. Analyze the logical structure of the following proof of the proposition ‘if \( n^2 \) is odd, then \( n \) is odd’, putting it (as given) into a skeletal proof diagram and identifying the rules of inference being used. Then criticize this proof from the point of view of simplicity and elegance. Finally, rewrite it in better form.*

**Proof:**

Suppose \( n^2 \) is odd.

In order to show that \( n \) is odd, we first note that either \( n \) is odd or it is even.

Suppose that \( n \) is even.

Then \( n = 2k \) for some integer \( k \).

But then \( n^2 = (2k)^2 = 4k^2 = 2 \cdot 2k^2 \).

Hence \( n^2 \) is even; but this contradicts what we are originally given.

Thus \( n \) must be odd, which was to be proved. ■

67. Prove that it is impossible to add or subtract the numbers 0 through 9 in any way, using each number exactly once, to arrive at the value 16.

68. Prove that \( n^2 + n + 1 \) is odd for all natural numbers \( n \). [Hint: divide the natural numbers up into some natural classification and prove the result for each type of number.]

69. Prove that if \( n \) and \( m \) are two integers, then either 4 divides \( m \cdot n \) or 4 divides neither \( m \) nor \( n \).
70. Prove that all positive integers have a prime factor. Pinpoint the places at which Int-Elim Rules for ∨ enter into the proof.

71. Locate and analyze a proof that there are infinitely many prime numbers (see Exercise 1.8-56) from the point of view of this lesson. Which rules of inference related to ‘or’ are used in it and where?

72. Using the result of Example 10, solve $A = \sqrt{2} + A$. Identify those rules of inference from SL for ∨ that are involved in solving this equation.

73. Using the result of Example 10, solve $x^2 - x - 6 = 0$, subject to the constraint that $x \geq 0$. Choose various letters to stand for the different sentences involved in your argument (provide a key!), and then symbolize your argument in terms of them. Make all logical connections between the various sentences as explicit as you can, whether or not you stated them in your original argument. Finally, identify which rules of inference from SL are involved in solving this equation.

74. Using the result of Example 10, find the solution set of all ordered pairs $(x, y)$ such that $x(1 - y^2) = 0$ and $(x+2)y = 0$ (see Exercise 1.2-45). Construct a step by step argument for your answer, noting where any Inference Rules are being used.

75. Show that the average of two rational numbers $a/b$ and $c/d$ lies between these two numbers. Choose various letters to stand for the different sentences involved in your argument (provide a key!), and then symbolize your argument in terms of them. Make all logical connections between the various sentences as explicit as you can, whether or not you stated them in your original argument. Finally, identify which rules of inference from SL are involved in your argument. You may use as known the fact that multiplication of an inequality by a positive number does not change the direction of the inequality.

76. **Absolute Value Proofs**
   Using Cases, where appropriate, prove for any real numbers $a$ and $b$ that the following results hold.
   a. $|-a| = |a|$
   b. $a \leq |a|$
   c. $-|a| \leq a$
   d. $|a + b| \leq |a| + |b|$

EC 77. **Rationals and Irrationals**
   Prove that there is a pair of irrational numbers $a$ and $b$ such that $a^b$ is rational.
   [Hint: consider what $\sqrt{2}^{\sqrt{2}}$ might be and use your conclusions to determine $a$ and $b$.]
HINTS TO STARRED EXERCISES 1.9

12. [No hint.]
13. [No hint.]
15. Use Cases in one direction and Add in the other.
18. An inference rule from this section is wrongly applied somewhere in the deduction.
19. Line 11 is problematic, but that’s not all.
33. Use CP for your main strategy, but inside the subproof, use an inference rule for ∨.
38. Use EO here.
45. EO works, but so does using some Replacement Rules.
53. Use EO in one direction and Cases in the other.
55. Protagoras likes the court’s decision if it favors him, but prefers his contract with Euathlus if the court’s ruling goes against him. Euathlus can argue similarly.
59. The problem provides its own hint.
64. Use a generalized version of DS for this. Make sure you nail all the crooks involved.
65. The first part proves an ‘or’ statement; the second part supposes an ‘or’ statement. Which inference rules are usually applied in these situations?