Symbolic Powers of Edge Ideals

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Disciplines
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Comments
Presentation at the 20th Biennial Conference of the Association for Christians in the Mathematical Sciences held at Redeemer College in Ancaster, Ontario, Canada on work that sprung out of a Kuyper Scholars Program project in Spring 2015 connecting algebra and graph theory.
Symbolic Powers of Edge Ideals

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Our project

Background: a student approached me to do an honors contract in a special topics course.

My research area: commutative algebra/algebraic geometry
Let $k$ be an algebraically closed field (e.g., $k = \mathbb{C}$).

We will primarily consider homogeneous ideals $I \subseteq R = k[x_0, x_1, \ldots, x_N]$. [The word \textit{form} is interchangeable with \textit{homogeneous polynomial}.]

Example

In $\mathbb{C}[X, Y, Z]$ such an ideal is $I = (XZ, YZ, X^3 - 3X^2Y - XY^2)$. A non-example is $J = (X^2 - Y, Z^2)$. 
Ordinary Powers

Given ideals \( I, J \subseteq R \), we may multiply ideals. Recall:

\[ IJ = (FG : F \in I, G \in J). \]

We may extend this to (ordinary) powers:

\[ I^r = (G_{i_1}G_{i_2} \cdots G_{i_r} : G_i \in I) \]

Example

Let \( I = (X, Y) \subseteq \mathbb{C}[X, Y, Z] \). Then \( I^2 = (X^2, XY, Y^2) \), \( I^3 = (X^3, X^2Y, XY^2, Y^3) \), etc.

Note: We have \( I^r \subseteq I^t \) if and only if \( r \geq t \).
Symbolic Powers

**Definition**

Given an ideal $I \subseteq R$, we define the $m$-th symbolic power of $I$ to be

$$I^{(m)} = R \cap \left( \bigcap_P (I^m R_P) \right).$$

This can reduce to a much cleaner definition if more information about $I$ is available.

**Note:** We have $I^{(r)} \subseteq I^{(t)}$ if and only if $r \geq t$. 
Ordinary vs. Symbolic

**Question**

*What is the relationship between $I^r$ and $I^{(m)}$?*

**Answer:** It depends on $I$.

A partial answer: $I^r \subseteq I^{(m)}$ if and only if $r \geq m$.

A (further) partial answer: $I^{(m)} \subseteq I^r$ implies $m \geq r$.

Before elaborating, we ask: what can symbolic powers look like?
Symbolic Powers of Edge Ideals
First studied by R. Villareal in the 1990s

Let $V = \{x_1, x_2, \ldots, x_n\}$ be a set of variables and consider the (simple) graph $G = (V, E)$, where $E$ contains 2-element sets comprised of pairs of the variables (so, e.g., $\{x_1, x_2\} \in E$ but $\{x_1, x_2, x_3\}, \{x_1^2\} \notin E$).

**Definition**

Given $G = (V, E)$ as above, the edge ideal of $G$ is $I(G) = \langle x_i x_j : \{x_i, x_j\} \in E \rangle \subseteq k[x_1, x_2, \ldots, x_n]$.

**Fact:** For an edge ideal $I$, $I^{(m)} = \bigcap_i P_i^m$, where the $P_i$ correspond to minimal vertex covers of $G$. 
\[ I = I(C_5) = (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_0) \]

Here, the ring is \( R = k[x_0, x_1, x_2, x_3, x_4] \), and the ideals corresponding to minimal vertex covers are \( P_1 = (x_0, x_1, x_3) \), \( P_2 = (x_0, x_2, x_3) \), \( P_3 = (x_0, x_2, x_4) \), \( P_4 = (x_1, x_2, x_4) \), \( P_5 = (x_1, x_3, x_4) \). Then
\[
I^{(2)} = P_1^2 \cap P_2^2 \cap P_3^2 \cap P_4^2 \cap P_5^2 \\
= (x_0^2x_1^2, x_0x_1^2x_2, x_1^2x_2^2, x_0x_1x_2x_3, x_1x_2^2x_3, x_2^2x_3^2, x_0^2x_1x_4, x_0x_1x_2x_4, \\
x_0x_1x_3x_4, x_0x_2x_3x_4, x_1x_2x_3x_4, x_2x_3^2x_4, x_0^2x_4^2, x_0x_3x_4^2, x_3^2x_4^2) \\
= I^2.
\]

But \( I^{(t)} \neq I^t \) for all \( t > 2 \).
Bipartite edge ideal characterization

Theorem (Simis-Vasconcelos-Villareal (1994))

Given an edge ideal $I = I(G) \subseteq k[x_1, x_2, \ldots, x_n]$ as above, the following are equivalent.

(i) $I^{(m)} = I^m$ for all $m \geq 1$.

(ii) The graph $G$ is bipartite.
A consequence of the previous theorem is: if $G$ is not bipartite and $I = I(G)$, then there exists a $t > 0$ such that $I^{(t)} \neq I^t$.

Our main question:

**Problem**

If $I = I(G)$ and $G$ is not bipartite, how do $I^{(m)}$ and $I^r$ compare?

**Problem (Invariant Problem)**

Compute invariants related to the containment $I^{(m)} \subseteq I^r$. 
A conjecture

Focus of the honors project at Dordt College in Spring 2015: what happens when $G$ is not bipartite?

**Conjecture (Ellis–Wilson–McLoud-Mann)**

Let $I = I(C_{2n+1}) \subseteq k[x_1, \ldots, x_{2n+1}]$ be the edge ideal of the odd cycle on $2n + 1$ vertices. Then

- $I^t = I^{(t)}$ for all $1 \leq t \leq n$;
- $I^t \neq I^{(t)}$ for all $t > n$. 

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Of importance when discussing ideal containments is the *initial degree*.

**Definition**

Let $J \subsetneq k[x_0, x_1, \ldots, x_N]$ be a nonzero homogeneous ideal. Define

$$\alpha(J) = \min \{ d : \text{there exists } 0 \neq f \in J, \deg(f) = d \}.$$ 

Note: if $\alpha(I^{(m)}) < \alpha(I^r)$ then $I^{(m)} \subsetneq I^r$.

**Example**

Given an edge ideal $I = I(G)$, $\alpha(I) = 2$ and $\alpha(I^r) = r\alpha(I) = 2r$.

Computing $\alpha(I^{(m)})$ is more delicate.

Given $I$, the edge ideal of $C_{2n+1}$,

$$\alpha(I^{(m)}) = 2m - \left\lfloor \frac{m}{n+1} \right\rfloor.$$
Proposition

Let \( I = I(C_{2n+1}) \subseteq k[x_1, \ldots, x_{2n+1}] \) be the edge ideal of the odd cycle on \( 2n + 1 \) vertices. Then \( I(t) \neq I^t \) for all \( t > n \).

Proof.

We know \( \alpha(I^t) = 2t \) and \( \alpha(I^{(t)}) = 2t - \lfloor \frac{t}{n+1} \rfloor \leq 2t - 1 < 2t \) when \( t > n \).

Our work attempting to prove the rest of the conjecture is ongoing.
Thanks

Thank you!