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Much More Than Symbolics: The Early History of Algebra and Its Significance for Introductory Algebra Education

Calvin Jongsma
Dordt College, calvin.jongsma@dordt.edu

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Much More Than Symbolics: The Early History of Algebra and Its Significance for Introductory Algebra Education

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Disciplines
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Comments
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Much More Than Symbolics:
The Early History of Algebra and Its Significance for Introductory Algebra Education

Calvin Jongsma
Dordt College

HPM East Coast Meeting
13 March 2011
Textbook Project Description in Brief

▲ Course for Prospective Middle School Mathematics Teachers
  ● Curricular Identity Crisis: Changing Educational Scene
  ● Refocused Content, Drawing from History of Mathematics
    - Focus on origins and historical development of key ideas
    - Global organization: mathematical topics
    - Internal sequencing: culture and chronology

▲ Need for Class Materials
  ● Extensive Outlines
  ● Homemade Exercises

▲ Textbook Production: Sections in Process
  ● Audience: Middle School Mathematics Teachers
  ● Focus on Educationally Significant Developments
  ● Historical Resources: Synthesize Current Research

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Educational Benefits for (Prospective) Mathematics Teachers

- **Cultural**
  - Acknowledge broader cultural matrix of mathematics
  - Appreciate multi-cultural contributions to mathematics
  - Contribute to integrated social studies units

- **Philosophical**
  - Acquire more realistic view of the nature of mathematics

- **Curricular & Pedagogical: Developmental Perspective**
  - Ontogeny recapitulates phylogeny?
  - Cognitive development, but with significant cultural differences
  - Recognize developmental prerequisites
  - Gain deepened understanding of meaning
  - Identify potential cognitive obstacles

- **Enrichment Activities**
  - Draw on alternative understandings and procedures
  - Incorporate supplementary activities
  - Explore simple primary source material
Table of Contents

1 Numeration Systems and Arithmetic
   1.1 Tally Marks, Roman Numerals, and the Abacus
   1.2 Egyptian Numeration and Arithmetic
   1.3 Mesopotamian Numeration and Arithmetic
   1.4 Greek Numeration and Arithmetic
   1.5 Chinese and Indian Numeration and Arithmetic
   1.6 Arabic and European Numeration and Arithmetic

2 Number Concepts
   1.1 Egyptian Unit Fraction Arithmetic
   1.2 Mesopotamian Sexagesimal Fraction Arithmetic
   1.3 Chinese Common Fraction Arithmetic
   1.4 Decimal Fractions
   1.5 Greek Fractions, Ratio, and Proportion
   1.6 Incommensurable Magnitudes and Irrational Numbers
   1.7 Negative Numbers
3 Measurement Geometry
   3.1 Flat-Sided Figures
   3.2 Curved Figures
   3.3 Pythagorean Theorem: Discovery and Use
   3.4 Similar Figures and Indirect Measurement

4 Deductive Geometry
   4.1 Rise of Deductive Reasoning
   4.2 Aristotelian Logic and Valid Arguments
   4.3 Logic and Mathematical Reasoning
   4.4 Greek Deductive Geometry
   4.5 Geometric Constructions and Congruent Triangles
   4.6 Theory of Parallels
   4.7 Area Comparisons and the Pythagorean Theorem
5 Algebraic Problem Solving

5.1 Early Algebra: Concrete and Simple Problem Solving
5.2 Linear Algebra: Solving Linear Systems
5.3 Quadratic Algebra: Algebra in Geometry Mode
5.4 Symbolic Algebra: Problem Solving with Symbols
5.5 Algebraic Geometry: Elementary Coordinate Graphing
5.6 Algebraic Structure: Generalized Arithmetic and Beyond

6 Combinatorics, Probability, and Statistics

(To Be Developed)
What is (Elementary) Algebra? Some Tacit Definitions

- **Popular & Technical Definitions**
  
  Algebra is a field that . . .
  
  - manipulates letters like numbers.
  - solves problems by doing symbolic calculations on equations.
  - generalizes arithmetic.
  - creates and explores abstract quantitative structures.

- **Definitions Put Forward in Educational Research**
  
  Algebra is a field that . . .
  
  - studies symbolic representation of problems using letters.
  - studies procedures for simplifying and solving equations.
  - studies general relationships among numbers.
  - studies theories of quantitative structures.
Definitions Adopted/Assumed by Histories of Mathematics
- Conventional periodization: Rhetorical, Syncopated, Symbolic
  - Assumes symbolism is central
  - Values developments for advancing symbolic representation
  - Treats early algebra as proto-algebra at best
  - Ironically, denies algebra-level status for Islamic contributions

Alternative Definitions Informed by History of Mathematics
- Open to features that transcend and contextualize notation
- Propose a semi-stable core with changing manifestations
  - Algebra is the premier problem-solving instrument.
  - Arithmetic calculates outputs; algebra solves for inputs.
  - Algebra treats unknowns in the same way as known quantities.
  - Algebra develops efficient systematic methods for problem solving.

Proper Focus for Elementary Algebra Education
- Focus less on symbolics at the outset
- Pay more attention to concrete problem-solving strategies
- Introduce symbolism as a powerful problem-solving strategy
Simple and Concrete Problem Solving

Some Sample Problems: Is This Arithmetic or Algebra?

- Problem 1
  A work supervisor has a crew of eight. He pays each person with three loaves of bread. How many loaves does he distribute?
  An arithmetic problem: $8 \times 3 = 24$.

- Problem 2
  A work supervisor receives 20 loaves of bread for his crew. He stores them on three equal-sized racks but has to leave two off. How many are on each rack?
  An algebra problem? $20 - 2 = 18; 18 \div 3 = 6$.

- Problem 3
  A surveyor measures a rectangular field, recording the area as 60 and the diagonals’ length as 13. What are the dimensions of the field?
  An algebra problem: Pythagorean triple $(5, 12, 13)$; or compute: $13^2 + 2 \times 60 = 289; 13^2 - 2 \times 60 = 49; (\sqrt{289} \pm \sqrt{49})/2 = 5, 12$. 

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Arithmetic Problem Solving for Unknown Input

- **Trial-and-Error Guessing**
  - Guess solution; adjust it up or down, depending on match-up

- **Method of False Position**: Systematic Guessing
  - Guess solution; scale it by (true output):(calculated output)
  - Exhibits understanding of problem’s proportional relationships
  - Used by various ancient cultures
  - Chinese *Method of Excess and Deficit*: Double False Position

- **Arithmetic-in-Reverse Solutions**
  - Solving problems by simple inverse calculations (e.g., division)
  - Indian *Method of Inversion*:
    reverse the calculation sequence, using inverse operations
    matches what we do with symbolic algebra
  - Example: three times a quantity plus four yields ten.
    \[ Q \rightarrow 3Q \rightarrow 3Q + 4 = 10; \quad 10 - 4 = 6; \quad 6 \div 3 = 2 \]
    \[ 3x + 4 = 10; \quad 3x = 10 - 4 = 6; \quad x = 6 \div 3 = 2 \]
Concrete and Geometric Modeling

- Egyptian *Aha* Problem: RMP #26
  A quantity plus its fourth-part gives 15. Find the quantity.

- Egyptian Solution: Algorithmic Calculation Sequence
  Calculate with 4: $4 + 1 = 5$.
  Divide: $15 \div 5 = 3$.
  Multiply: $4 \times 3 = 12$, the quantity.

- Egyptian Solution: *Method of Part-Size*
  Make a fourth-part the unit: $4 + 1$ parts.
  5 parts are 15; 1 part is 3.
  The full quantity is 4 times as much: 12.

- *Method of False Position*
  Guess 4: $4 + 1 = 5$; $15 \div 5 = 3$; $3 \times 4 = 12$
  Not exactly the Egyptian scribe’s solution: bad guess; wrong final multiplication; counterfactual vs. concrete reasoning.
• Is the Egyptian *Method of Part-Size* Algebraic? Yes.
  - It’s a systematic method of problem solving.
  - It solves for an unknown input.
  - It treats the unknown input as a quantity one can calculate with.

• Algebraic features of the Egyptian *Method of Part-Size*
  - It essentially introduces a new unknown (a fourth-part) and so reconfigures the problem to be solved.
  - It concretely matches a symbolic solution method we might use.

\[
x + \frac{x}{4} = 15; \quad 4 \left(\frac{x}{4}\right) + \left(\frac{x}{4}\right) = 15; \quad 5 \left(\frac{x}{4}\right) = 15
\]

\[
\left(\frac{x}{4}\right) = 15 \div 5 = 3; \quad x = 4 \left(\frac{x}{4}\right) = 4 \times 3 = 12
\]
Old Babylonian Linear Problem: TMS VII
The length of a rectangle plus one-fourth its width is 7. The length plus the width is 10. Find the length and width.

Babylonian Solution: Algorithmic Calculation Sequence
(1) Multiply: $4 \times 7 = 28$.
(2) Subtract: $28 - 10 = 18$.
(3) Divide: $18 \div 3 = 6$, the length.
(4) Subtract: $10 - 6 = 4$, the width.

Concrete Interpretation; Associated Symbolic Algebra
(0) Length plus one-fourth width is 7; $L + \frac{1}{4}W = 7$
Length plus width is 10. $L + W = 10$
(1) Scale: four lengths plus one width is 28. $4L + W = 28$
(2) Subtract: three lengths is 18. $3L = 18$
(3) Divide: one length is 6. $L = 6$
(4) Subtract: one width is 4. $W = 4$

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• Babylonian Solution’s Underlying Geometric Reasoning

(0) Length plus one-fourth width is 7;
\[ L + \frac{1}{4}W = 7 \]

(1) Scale: four lengths plus one width is 28.
\[ L + L + L + L + W = 28 \]

(0) Length plus width is 10.
\[ L + W = 10 \]

(2) Subtract: three lengths is 18.
\[ L + L + L = 18 \]

(3) Divide: one length is 6.
\[ L = 6 \]

(4) Subtract: one width is 4.
\[ W = 4 \]
• Is the Babylonian Solution Method Algebraic? Yes.
  - It’s a systematic method of problem solving.
  - It solves for unknown inputs.
  - It treats unknown inputs as quantities one can calculate with.

• Algebraic features of the Babylonian Solution Method
  - It doesn’t solve for one unknown and substitute to reduce the data
to one complex equation in one unknown.
  - It arithmetically combines equations and eliminates unknowns.
  - It concretely matches a symbolic linear algebra solution method.

\[
\begin{align*}
L + \frac{W}{4} &= 7 & 4L + W &= 28 \\
L + W &= 10 & L + W &= 10 \\
3L &= 18 & L &= 6 \\
W &= 4
\end{align*}
\]
Geometric Quadratic Algebra

- Babylonian Quadratic Algebra
  - Babylonian Quadratic Problem: YBC 6967
    A reciprocal exceeds its reciprocal by 7. Find the two numbers.
    [Note: the product of a reciprocal pair is taken as 60.]
  - Babylonian Solution: Algorithmic Calculation Sequence
    (1) Halve: \( 7 \div 2 = 3 \frac{1}{2} \).
    (2) Square: \( 3 \frac{1}{2} \times 3 \frac{1}{2} = 12 \frac{1}{4} \).
    (3) Add: \( 12 \frac{1}{4} + 60 = 72 \frac{1}{4} \).
    (4) Take the square root: \( \sqrt{72 \frac{1}{4}} = 8 \frac{1}{2} \).
    (5) Add and subtract: \( 8 \frac{1}{2} + 3 \frac{1}{2} = 12, \ 8 \frac{1}{2} - 3 \frac{1}{2} = 5. \)
    These are the numbers sought: \( 12 \times 5 = 60, \ 12 - 5 = 7. \)
Babylonian Solution’s Underlying Geometry

\[
\begin{align*}
X - Y &= 7 \\
3 \frac{1}{2} \times 3 \frac{1}{2} &= 12 \frac{1}{4} \\
\sqrt{72 \frac{1}{4}} &= 8 \frac{1}{2} \\
7 \div 2 &= 3 \frac{1}{2} \\
12 \frac{1}{4} + 60 &= 72 \frac{1}{4} \\
8 \frac{1}{2} \pm 3 \frac{1}{2} &= 12, 5
\end{align*}
\]
Symbolic Formulation of Algorithm

(0) \( XY = P; \quad X - Y = D. \)

Combining by substituting, \( X(X - D) = X^2 - DX = P. \)

(1) \( D \div 2 = \frac{D}{2}. \)

(2) \( \left( \frac{D}{2} \right)^2 = \frac{D^2}{4}. \)

(3) \( \frac{D^2}{4} + P \)

(4) \( \sqrt{\frac{D^2}{4} + P} \)

(5) \( X, Y = \sqrt{\frac{D^2}{4} + P} \pm \frac{D}{2} \): a version of the Quadratic Formula.
Are Babylonian Quadratic Solution Methods Algebraic? Yes.
- They give systematic methods of problem solving.
- They solve for unknown inputs.
- They treat unknown inputs as quantities one can calculate with.

Algebraic features of Babylonian Quadratic Solution Methods
- They solve problems using effective geometric cut-and-paste complete-the-square methods.
- They concretely match symbolic algebraic solution methods.
- Their methods do not give a version of the quadratic formula; instead, they give the equivalent of our complete-the-square procedure.
- The Babylonian Complete-The-Square Method is historically very important: this is what converted algebra from mere recreational problem solving into the art and science of problem solving.
Islamic Quadratic Algebra

- Islamic Quadratic Problem: al-Khwārizmī, Type 4 Equation
  Squares and roots equal to numbers: one square and ten of its roots equals thirty-nine.

- al-Khwārizmī’s Solution: Algorithmic Procedure (all in words)
  1. Halve the number of roots: five.
  2. Multiply this by itself: twenty-five.
  3. Add this to thirty-nine: sixty-four.
  4. Take this number’s root: eight.
  5. Subtract half the original roots: three.
  6. Three is the root of the square sought; the square itself is nine.
**Geometric Demonstration of Solution**

Draw a square, and add rectangles of length 5 to two sides of the square. Total area, as given, is 39.

Complete the square; 25 is added to the area, giving 64. The side of the large square is 8; the small square’s side is 5 less: 3.

\[
\begin{align*}
\text{Area} & = x^2 + 10x = 39 \\
\text{Complete the square:} & \quad x^2 + 10x + 25 = 39 + 25 = 64 \\
\text{Solve for } x : & \quad (x + 5)^2 = 8^2 \\
\text{Simplify:} & \quad x + 5 = 8 \\
\text{Solve for } x & \quad x = 8 - 5 = 3
\end{align*}
\]
Symbolic Formulation of Algorithm

(0) \( x^2 + bx = c \)

(1) \( b \div 2 = \frac{b}{2} \)

(2) \( \left( \frac{b}{2} \right)^2 = \frac{b^2}{4} \)

(3) \( \frac{b^2}{4} + c \)

(4) \( \sqrt{\frac{b^2}{4} + c} \)

(5) \( x = \sqrt{\frac{b^2}{4} + c} - \frac{b}{2} : \) a version of the Quadratic Formula.
• Are Islamic Quadratic Solution Methods Algebraic? Yes.
  - They systematically solve a wide variety of elementary problems.
  - They solve for unknown inputs.
  - They treat unknown inputs as quantities one can calculate with.
  - They perform calculations with algebraic expressions and manipulate equations, albeit verbally.

• Algebraic features of Islamic Quadratic Solution Methods
  - Elementary algebra is systematically organized into a discipline: equations are fully enumerated and categorized, effective solution procedures are developed and their legitimacy geometrically demonstrated, algebraic calculations are explored, and applications are made.
  - Their solution algorithms match symbolic algebraic solution procedures.
  - However, solution methods are not yet fully uniform, nor do they generate formulas.
  - Islamic algebra becomes the springboard for further developments in later European circles.
Summary: Educational Lessons from HoM

► Norm of Concrete/Holistic Beginnings
  • Begin concretely, keeping the problem-solving goal of algebra centrally in mind
  • Model problems appropriately, using a variety of concrete approaches and effective procedures

► Norm of Progressive Development
  • Use more complex models and procedures as needed and as students are ready for them
  • Introduce symbolic abstraction and operations gradually, in parallel with concrete representation and manipulations

► Norm of Seeking Efficient Uniform Procedures (Later Talk)
  • Reveal the limitations of a narrowly concrete approach
  • Demonstrate the power and simplicity of an abstract symbolic approach
Value of History of Mathematics for Learning Algebra
- Suggests ways to highlight/explain key ideas, avoid difficulties
- Offers both curricular and pedagogical insights
- Provides enrichment and exploratory materials

Questions or Suggestions?