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Al-Khwarizmi: Founder of Classical Algebra

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Abstract
Adopting a historically defensible definition of "algebra," we will begin by exploring a few examples of algebra prior to al-Khwarizmi. We will then examine what algebra became through al-Khwarizmi's work. In conclusion, we will assess the historical importance of al-Khwarizmi's contributions for developments in European algebra.

Keywords
Al-Khwarizmi, algebra, Babylonian algebra, medieval Arabic algebra, history of mathematics

Disciplines
Algebra | Christianity

Comments
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Founder of Classical Algebra

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OUTLINE

✠ What is Algebra? Some Tacit Definitions

- Popular Perceptions

  Algebra is a branch of mathematics that . . .
  - calculates with letters as if they were numbers.
  - manipulates symbolic equations to solve them.
  - provides formulas for science; generalizes arithmetic.
  - causes confusion, frustration, and grief for beginning students.

- Mathematics Educators' Definitions

  Algebra is a branch of mathematics that . . .
  - provides symbolic representations of problems, using letters.
  - develops procedures for transforming and solving equations.
  - symbolizes general relationships/laws for numbers.
  - studies abstract structures (groups, fields; lattices; . . .).

- Outlook Assumed by (Older) Histories of Mathematics
  - Conventional periodization: Rhetorical, Syncopated, Symbolic
    - Assumes symbolic form is central/definitive
    - Values developments for advancing toward representation via letters
    - Treats early algebra as proto-algebra at best
    - Ironically, denies full-algebra status for Arabic contributions
- Outlook Offered by Recent Educational Research
  - Symbolization perspective
    - Move beyond narrow syntactic concerns
    - Focus on symbolization process (representations, transformations)
    - Symbols permit mechanization and further abstraction.
  - Generalization focus
    - Algebra offers a way to generalize/make general statements.
    - Algebra reasons with general claims, within a system of symbols, according to syntactic rules.

- Refined Definition Informed by History of Mathematics
  - Algebraic features transcend and contextualize symbolization
    - Algebra is the premier quantitative problem-solving instrument.
      - Arithmetic calculates outputs; algebra solves for unknown inputs.
      - Algebra treats unknown quantities the same as known quantities.
      - Algebra uses efficient, systematic problem-solving methods.
    - Algebra enables mathematical modeling. (not our focus)
    - Algebra studies abstract structures. (also not our focus)

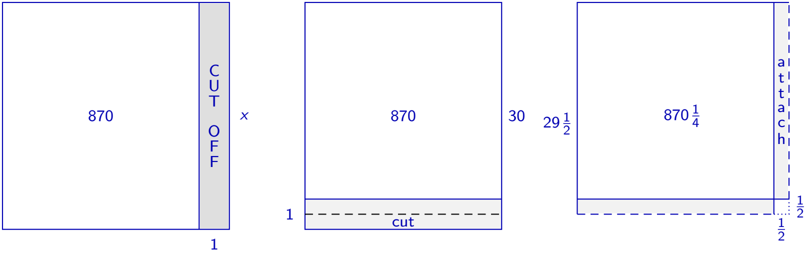
- History of Algebra Can Serve Mathematics Education
  - It can suggest ideas for teaching and learning algebra.
  - It can offer study materials for exploring/enriching algebra.
  - It can help refine our concept of algebra for educational research.

Babylonian Algebra

- Historiographic Details
  - Sources: half a million cuneiform tablets to decipher/synthesize
  - Early 20th century history of mathematics
    - Mathematics studied largely in isolation from culture
    - Algebra seen as having attained a moderate level of abstractness
  - Late 20th century history of mathematics
    - Mathematical developments now related to cultural contexts
    - Algebraic developments interpreted using more careful analysis
- Cultural and Mathematical Context (1800 BC)
  - Mesopotamian scribes: administrators and teachers
  - Computational expertise: sexagesimal place-value arithmetic
  - Algebraic problems: arising out of a surveyors' riddle tradition?
  - Algebraic solutions: terms suggest geometric substratum

- Babylonian Quadratic Algebra: Problem and Solution
  - Babylonian quadratic problem BM 13901 #2
    I subtracted the side of a square from its area; it was 14,30 [870]. Find the square's side.
  - Babylonian solution: algorithmic calculation sequence
    (1) Halve: $1 \div 2 = \frac{1}{2}$.
    (2) Square: $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.
    (3) Add: $870 + \frac{1}{4} = 870\frac{1}{4}$.
    (4) Take the square-side: $\sqrt{870\frac{1}{4}} = 29\frac{1}{2}$.
    (5) Add the sides: $29\frac{1}{2} + \frac{1}{2} = 30$.
    The original square's side is 30.

- Babylonian solution's geometry: cut-and-paste



$$x^2 - x = 870$$

$$x^2 - x + \frac{1}{4} = 870\frac{1}{4}$$

$$x - \frac{1}{2} = \sqrt{870\frac{1}{4}} = 29\frac{1}{2}$$

$$x = 29\frac{1}{2} + \frac{1}{2} = 30$$

- Are Babylonian quadratic solution methods algebraic? *YES!*
  - They solve for unknown inputs.
  - They calculate with unknown inputs like ordinary quantities.
  - They provide systematic methods of problem solving.

- Main features of Babylonian algebraic problem solving
  - Algorithmic step-by-step solution process; no formulas
  - Geometric medium, employing dynamic cut-and-paste methods
  - Geometric transformations match symbolic solutions
  - *Completing-the-Square* – versatile procedure – gave birth to algebra: recreational problem solving became the art of problem solving

Arabic Algebra

- Cultural and Mathematical Context
  - Spread of the Arabic Empire (630 – 730; Spain to India)
  - Growing familiarity with other cultures' literature, philosophy, science, mathematics, etc.
  - House of Wisdom established (Baghdad, 825); intellectual center for translation, scholarship, and scientific research
  - Arabic algebra: probably indigenous origins; later Greek influence
  - al-Khwārizmī's mathematical texts
    - Arithmetic text introducing Indian numerals and reckoning (825)
    - Kitāb al-jabr w'al-muqābala (Calculation by al-Jabr and al-Muqābala): the founding Arabic text on algebra (830)
    - al-Khwārizmī's texts very important for European mathematics

- Contents and Organization of al-Khwārizmī's Algebra
  - Doxological dedicatory preface (2 pages)
  - Systematic treatment of equations, all done in words (45 pages)
    - Six standard equation types identified (combinatorial criteria)
    - Standard equation types solved algorithmically
    - Standard solution procedures justified/illustrated geometrically
    - Computing with algebraic expressions and radicals
    - More complex equations (6 examples, 34 problems) reduced to standard types for solution by verbal transformations
  - Algebraic applications (no genuine quadratic solutions)
    - Commercial problems: Rule of Three (2 pages)
    - Measurement problems: area, volume calculations (15 pages)
    - Islamic inheritance problems (110 pages)

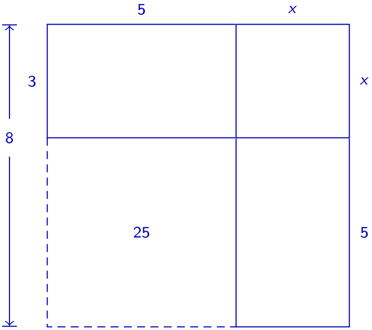
- Types of Quantities Compared in al-Khwārizmī's Algebra
  - Numbers
  - Roots/Sides
  - Squares

- al-Khwārizmī's Standard Equation Types
  - Simple types
    (1) Squares equal to roots [$ax^2 = bx$]
    (2) Squares equal to numbers [$ax^2 = c$]
    (3) Roots equal to numbers [$bx = c$]
  - Compound types
    (4) Squares and roots equal to numbers [$ax^2 + bx = c$]
    (5) Squares and numbers equal to roots [$ax^2 + c = bx$]
    (6) Roots and numbers equal to squares [$bx + c = ax^2$]

- al-Khwārizmī's Algebra: Some Problems and Solutions
  - Type 4 equation: squares and roots equal to numbers
    One square and ten of its roots equals thirty-nine. Find the root and the square.
  - al-Khwārizmī's algorithmic solution procedure (all in words):
    (1) Halve the number of roots: five.
    (2) Multiply this by itself: twenty-five.
    (3) Add this to thirty-nine: sixty-four.
    (4) Take this number's root: eight.
    (5) Subtract half the original roots: three.

    Three is the root of the square sought; the square is nine.

- Geometric demonstration of the solution



Draw a square,
and add rectangles of length 5
to two sides of the square.
Total area, as given, is 39.
Complete the square:
25 is added to the area, giving 64.
The side of the large square is 8;
the small square's side is 5 less: 3.

$$x^2 + 10x = 39$$
$$x^2 + 10x + 25 = 39 + 25 = 64$$
$$(x+5)^2 = 8^2$$
$$x + 5 = 8$$
$$x = 8 - 5 = 3$$

- Fully symbolic formulation of solution algorithm

(0) $x^2 + bx = c$

(1) $b \div 2 = \frac{b}{2}$

(2) $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$

(3) $\frac{b^2}{4} + c$

(4) $\sqrt{\frac{b^2}{4} + c}$

(5) $x = \sqrt{\frac{b^2}{4} + c} - \frac{b}{2}$: a version of the Quadratic Formula.

- Type 5 equation: squares and numbers equal to roots
  One square and twenty-one equals ten roots. Find the square.

- al-Khwārizmī's algorithmic solution procedure:
  (1) Halve the number of roots: five.
  (2) Multiply this by itself: twenty-five.
  (3) Subtract twenty-one from this: four.
  (4) Extract the root: two.
  (5) Subtract this from/add this to half the roots: three/seven.

  Three/seven is a root of such a square, which is nine/forty-nine; for each solution, one square plus twenty-one equals ten roots.

- al-Khwārizmī's geometric justifications for type 5 and 6 equations: more complex figures

- Sample problem 5 in al-Khwārizmī's Algebra

  I divided ten into two parts. Multiplying each part by itself and adding these products together, the sum was fifty-eight. Find the two parts.

al-Khwārizmī's Solution

Let one of the parts be a thing and the other ten minus that thing. Multiply ten minus a thing by itself: it is one hundred and a square minus twenty things. Also multiply a thing by a thing; it is a square.

The sum of these products is a hundred plus two squares minus twenty things, which equals fifty-eight.

Now take the twenty negative things from the hundred and the two squares and add them to fifty-eight; then a hundred, plus two squares are equal to fifty-eight and twenty things [done by *al-jabr*].

Modern Symbolic Counterpart

Let x, $10 - x$ be the two parts.

Then $(10 - x)^2 = 100 + x^2 - 20x$.

So $(10-x)^2 + x^2 = 100 + 2x^2 - 20x = 58$.

Thus, $100 + 2x^2 = 58 + 20x$.

Reduce this to a square by taking half of everything. It then becomes: fifty and a square are equal to twenty-nine and ten things.

Halving, $50 + x^2 = 29 + 10x$.

Reduce this [*al-muqābala*], by taking twenty-nine from fifty; there remains twenty-one and a square equal to ten things.

Subtracting/combining like terms, $21 + x^2 = 10x$.

[Now al-Khwārizmī starts the standard solution procedure for a Case 5 equation.]

Halve the number of roots; it is five.

$10 \div 2 = 5$

Multiply this by itself (twenty-five) and subtract twenty-one; four remains.

$5^2 = 25$

$25 - 21 = 4$

Extract the root, it is two.

$\sqrt{4} = 2$

Subtract this from half the number of roots (from five); there remains three. This is one part.

$5 - 2 = \underline{3}$

The other is seven, the root added to half the number of roots.

$5 + 2 = \underline{7}$

- Are al-Khwārizmī's solution methods algebraic? *YES!*
  - They solve for unknown inputs.
  - They calculate with unknown inputs as with numbers.
  - They systematically and efficiently solve equations.

- Algebraic features of Arabic solution methods
  - Equations are categorized, and canonical forms are identified.
  - Solutions are algorithmically found by operating on known and unknown quantities.
  - Solution procedures are geometrically demonstrated.
  - Algebraic expressions are computed, and equations are manipulated, to reduce equations to canonical form – albeit verbally.
  - Solution methods match symbolic algebra solution procedures. However, they are not yet fully uniform, nor do they give formulas.
  - Nevertheless, algebra is now systematically organized into a discipline, a science or theory of equations
  - Arabic algebra becomes the springboard for further developments in later European circles.

✠ Summary: Educational Lessons from HoAlg

- Norm of Concrete/Holistic Beginnings
  - Begin concretely, keeping the central problem-solving goal of algebra in mind
  - Model problems appropriately, using a variety of concrete approaches and effective procedures, including geometric ones
- Norm of Progressive Comprehension
  - Use more complex models and procedures as needed and as students are ready to use them
  - Introduce symbolic abstraction and operations gradually, in parallel with concrete representation and manipulations
- Norm of Efficient Uniform Procedures (future talk)
  - Reveal the limitations of a narrowly concrete approach (homogeneity, dimensionality, positivity)
  - Demonstrate the power and simplicity of an even more systematic/uniform abstract symbolic approach

- Value of History of Mathematics for Learning Algebra
  - Offers curricular and pedagogical insights to teachers
    - Suggests ways to highlight/connect/explain key ideas
    - Suggests ways to avoid difficulties
  - Provides enrichment and exploratory materials for students