3-21-2016

Philosophical Introduction to Set Theory (Book Review)

Calvin Jongsma
Dordt College, calvin.jongsma@dordt.edu

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Abstract

Reviewed Title: *Philosophical Introduction to Set Theory* by Stephen Pollard. Dover Publications, 2015, 180 pp. ISBN: 9780486797144.

Keywords

book review, Philosophical Introduction to Set Theory, Stephen Pollard

Disciplines

Christianity

Comments

Access book review from publisher's site:

http://www.maa.org/press/maa-reviews/philosophical-introduction-to-set-theory

Philosophical Introduction to Set Theory

In the 1960s, progressive mathematics educators in the United States decided that elementary mathematics should be taught in a more conceptual fashion to accelerate the production of scientists and engineers able to meet the Soviet challenge in the Space Race. The curricular changes proposed by the "New Math" revolution introduced set theory early on (objects gathered into collections, one-to-one matching) as a basis for learning arithmetic, and they emphasized the importance of algebraic structure (commutativity, the distributive property). In hindsight, this may seem a somewhat peculiar and misguided strategy, but educators were drawing upon what they thought were the most modern views of the nature of mathematics as revealed during the first half of the 20th century by important foundational developments in set theory, logic, and abstract algebra.

Stephen Pollard's 1990 *Philosophical Introduction to Set Theory*, republished 25 years later in an inexpensive Dover paperback edition, shows no sympathy for connecting school mathematics to set theory; its emphasis goes in the other direction. Pollard repudiates the idea that being familiar with everyday collections, as promoted by New Math educators, is (or even can be) of genuine assistance in learning set theory, and he takes some pains to refute this myth. The concept of a mathematical set, according to Pollard, can only be obtained by studying set theory itself. One implicitly learns what a set is by working with the axioms, definitions, and theorems of set theory and by learning what the deductive limitations of various axiomatizations are.

Pollard asserts (Chapter III) that everyday collections differ from mathematical sets in several key respects. One difference is whether the empty set should be considered a collection (since nothing is being collected); this problem parallels one mathematicians faced earlier in deciding whether 0 was a number. A more crucial issue, Pollard argues at some length, is the fact that commonsense collection usage does not allow collections themselves to be taken as members of higher-level collections. This is certainly true for the most part, for such constructions are rarely needed in everyday life — though baseball fans will point out that teams are members of leagues, not their players, so there is at least a simple commonplace instance where a set of sets occurs. Furthermore, it took logicians and mathematicians themselves some time to distinguish between set membership and class inclusion, as well as between single elements and singletons, but this doesn't mean that they weren't dealing with some version of a set concept prior to when set theory received its first axiomatization by Zermelo in 1908.

Besides arguing that everyday collections are different in nature from mathematical sets, Pollard argues that history of mathematics supports his perspective. Set theory did not evolve from considering pluralities (the Greek notion of number), he says, but arose within 19th century real analysis. And here it was not due to a concern with finite collections but with various infinite point sets that Cantor explored as he investigated the convergence of Fourier series. Cantor's discovery of a mathematical purpose for distinguishing different types of infinite sets is what led to his developing a theory of (transfinite) sets.

This contextualization of set theory is certainly true, but Pollard spends an entire chapter (Chapter II) exploring the back history of analysis — dipping into the Middle Ages (the Oxford

Calculators and Oresme), looking at the rise of symbolic algebra (Viète) and analytic geometry (Descartes), touching on 17th, 18th, and 19th century debates over the nature of a function — before finally arriving at how set theory developed out of Cantor's research in analysis. Like me, the reader may wonder whether all this earlier history is pertinent, but I appreciate the recognition that philosophizing shouldn't be done in a historical vacuum. Interestingly enough, this seems to be the way others are also doing philosophy of set theory (cf. Mary Tiles in her 1989 *The Philosophy of Set Theory*, and Penelope Maddy in her 2011 *Defending the Axioms: On the Philosophical Foundations of Set Theory*).

Pollard sees set theory as the reigning foundation of mathematics (and thus an essential topic for any philosophy of mathematics), though he briefly discusses the more recent challenge of category theory for this role, admitting that a time may eventually come when set theory's hegemony is broken. His reasons for sticking with set theory are mainly that it has provided fertile conceptual tools and terminology for a wide variety of subfields of mathematics, unifying a very diverse theoretical landscape, and that it provides a central theory in which mainstream theories of mathematics can be nicely interpreted (set-theoretic reductionism).

Pollard expounds his position on set theory by devoting roughly equal time to philosophical and technical matters. In the latter vein, chapters IV and VII provide logical interludes that discuss the notions of theory interpretability and the relation of second order logic and plural quantification to set theory respectively; chapter VI considers some metamathematics associated with the membership relation and the successor function; and chapter VIII, augmented by two appendices, explores the structures of iterative hierarchies for set theory. His discussion of various philosophical issues and outlooks inevitably expands on some of these technical matters as well.

Using some ideas of Michael Dummett, chapter V presents an extended argument in favor of adopting a formalist outlook on set theory. Oversimplifying this somewhat, having rejected any pre-theoretical or realistic Platonic basis for understanding mathematical sets or set membership, Pollard concludes that our knowledge of sets derives solely from our deductive development of the axioms and definitions of set theory and from discovering the deductive constraints (incompleteness and undecidability results) on such theories. In fact, the proper tasks of a set theorist are encapsulated in these deductive activities; the structures of interest to the set theorist are the theories themselves, not any supposed universe of sets existing independently of our theorizing. Cantor's Continuum Hypothesis, therefore, has no objective truth value outside of some axiomatic theory about large sets.

While Pollard believes he has presented formalism as "a genuinely compelling philosophical outlook," he realizes that this outlook is not accepted by everyone doing research into set theory nor by mathematicians in general. He therefore concludes his book in chapter IX by briefly sketching the outlines of an alternative, structuralist philosophy of set theory, which may better capture the thinking of practicing mathematicians. Here Pollard draws upon the work of Michael Resnik in particular. Since a mathematical theory is unable to distinguish between its isomorphic models, we should consider such theories, he says, as studying the abstract structure of all such models (a notion that fits well with considering category theory as foundational). The objects being studied should be thought of as abstract entities within such a structure, whose features are

only those specified within the theory. Sets should thus be considered as positions within the set-theoretic hierarchy axiomatized by some theory, and nothing more.

The reader can decide whether such an alternative viewpoint truly provides a viable counter to the formalism Pollard espoused earlier in the book (one may now take a realist view of abstract structures, if not the entities they contain). Pollard recognizes in the concluding paragraph that this perspective needs to be fleshed out further, but he believes it holds promise for solving problems in the philosophy of set theory in particular and mathematics in general.