Axiomatic Structure and the Method of Analysis: Shifting Styles in the History of Mathematics

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Comments
AXIOMATIC STRUCTURE AND THE METHOD OF ANALYSIS:
SHIFTING STYLES IN THE HISTORY OF MATHEMATICS

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Twentieth century mathematicians are heirs of Hilbert's legacy. Mathematicians have universally accepted the axiomatic method as the true mode of procedure in mathematics. They naturally permit themselves to use intuitive ideas and heuristic arguments to discover new results, but once these results are found, they must be put within an axiomatic framework. If it seems that they cannot be deduced from what is already known, mathematicians investigate whether they are consistent with known results, and, if so, whether they are independent of them. This axiomatic approach to mathematics can found in number theory, algebra, topology, or indeed anywhere.

This description naturally paints mathematical practice with a pretty broad brush, so it is bound to hide some discrepancies. Some mathematicians oppose a formal, axiomatic approach to mathematics on philosophical grounds, while others feel this ideal may be relaxed somewhat, especially within education. But despite these exceptions, I think my caricature describes an undercurrent that is still strong in the mathematical community. Legitimate mathematics the world over is mathematics that can be or that has been incorporated into an acceptable axiomatic system.

Strange as it may seem, this has not always been so. As a universally accepted code of practice, it is a phenomenon of twentieth century mathematics, though as an ideal, its roots stretch back into ancient Greek times. The mathematical practice of the seventeenth and eighteenth centuries provides us with a historical example of another way of approaching mathematics, the way of analysis. In my talk tonight, I want to focus on the rise of this analytic movement. By way of contrast and introduction I will first describe the Greeks' ideas about mathematics as a demonstrative science and the meaning and function of analysis within this school of thought. After looking at seventeenth century analysis in some detail, I will conclude with a few brief remarks on some nineteenth century trends which contributed to the rise of the abstract, axiomatic approach of the twentieth century.

GREEK MATHEMATICS AND THE IDEAL OF AXIOMATIZATION

It seems fairly certain that mathematics as a demonstrative science
is the product of ancient Greece. The term "Greek mathematics" brings axiomatic or synthetic Euclidean geometry immediately to mind. However, since the work of Otto Neugebauer on Babylonian astronomy and mathematics, historians of science have been alerted to another strand of mathematics in Greek culture. Contemporaneous with the deductive, geometrical mathematics with which we are all well acquainted, there was a more practice-oriented mathematics which was computational, algorithmic, and approximative in nature. This type of mathematics probably derived from Babylonian mathematical practice, and manifested itself particularly in the later astronomical work of the time. I will not have much to say about this trend, but I should note that its existence in later Hellenistic times was often interpreted by older historians of mathematics as part and parcel of the decay of mathematics after the time of Archimedes and Appolonius.

Realizing that axiomatic mathematics was not all that was done in Greek times, it is nevertheless that style of mathematics in which we are particularly interested here. Approximately a century and a half before Euclid wrote his Elements, Hippocrates of Chios is reported to have written the first systematic deductive treatise on geometry. His "elements" was followed by others, and the tradition was still thriving in Aristotle's time, for he speaks as if his reader is familiar with such a treatment. Aristotle's analysis of the elements which go into making up a demonstrative science is no doubt based upon this tradition. We can therefore learn something about this deductive approach by looking at Aristotle.

True knowledge, for Aristotle, is knowledge of why a thing is what it is. This can only be had when one traces the necessary causes of a thing back to its first or ultimate principles. First principles are primary truths which command our belief and are themselves indemonstrable. They must be accepted purely on the basis of experience and scientific intuition. Upon their foundation, all other truths may be demonstrated (i.e., proved to be true) by means of syllogistic reasoning.

The demonstrative method thus functions for Aristotle as the means for gaining true and certain knowledge of the world, regardless of the subject matter. Though Aristotle does not proceed axiomatically himself (demonstration is not a valid tool in philosophical discussions because philosophy lacks a specific subject matter), yet he has formulated an approach to knowledge whose scope extends far beyond mathematics. As such, Aristotle's "logic" was to wield great influence upon later thinking about the proper scientific method, both within mathematics and without.

Mathematics, for Aristotle, deals with quantity, that which can be divided into parts. There are two irreducible types of quantity: number, which is discrete quantity, and magnitude, which is contin-
uous quantity. Corresponding to these sub-categories there are two main fields of mathematics: arithmetic and geometry. Each of these is organized axiomatically. Mathematical results are proved by prior results and are ultimately based upon the definitions and first principles that are set out without proof.

Definitions, such as "a point is that which is indivisible and has position," or "number is a plurality of units" explain the essential nature of the terms mentioned and only require to be understood. Definitions tell how words are being used; they do not assert the existence of the terms' referents. Geometry and arithmetic take the existence of their subject matter for granted, either explicitly or implicitly, but objects having certain properties must either be posited or proved to exist. Thus, points and lines are assumed to exist for geometry and numbers for arithmetic, but the existence of things like equilateral triangles or prime numbers must be proved.

In the category of quantity, first principles are of two types: those which are common to both arithmetic and geometry, like "equals subtracted from equals leaves equals," and those which especially belong to each (Aristotle gives no examples here, but we can appropriate one of Euclid's for illustration: "a circle can be described with any center and any radius")/ Aristotle is quite emphatic that demonstration can never pass out of one subject area into another, so he forbids one to prove geometric theorems by means of arithmetic, and vice versa. Those principles which are shared by arithmetic and geometry, therefore, are common to both only in a formal, analogical sense. Strictly speaking, they can only be used in their arithmetic or geometric senses, and not as general principles.

We will gauge Aristotle's importance for the development of mathematical methodology by looking at the structure of Euclid's Elements. For it is there that the axiomatic method discussed by Aristotle found a home which was to become the model of exact thought for mathematics as well as science in general for over 2000 years.

Euclid wrote his Elements about 300 B.C., approximately one generation after the death of Aristotle. Drawing heavily upon the work of previous writers on arithmetic and geometry, Euclid composed an elementary mathematics textbook. Its results and their proofs were not entirely original, but Euclid undoubtedly added improvements of his own as he organized the various source materials into a systematic and coherent treatise. Because it presented a wide variety of theories and topics in a masterful, deductive fashion, it immediately became a classic, making its predecessors superfluous.

The text begins by listing twenty-three definitions, in which such terms as "line," "plane angle," "circle," and "equilateral triangle" are defined. Following this, there are five "postulates" which assert,
among other things, the possibility of performing certain constructions. Finally, five "common notions" are given, including such statements as "if equals be subtracted from equals, the remainders are equal" and "the whole is greater than the part." After having laid the necessary foundation for his work, Euclid proceeds to prove his propositions, starting with the theorem that shows how to construct an equilateral triangle.

Comparing Euclid's axiomatic practice with Aristotle's theory of demonstration, we note the following similarities. First of all, Euclid's definitions agree fairly well with what Aristotle says about scientific definitions. In particular, they state the essential meaning of the terms, but they do not assert existence. That is done separately, either by a first principle (e.g., by the postulate about constructing a circle) or by a theorem (e.g., by the proposition which demonstrates the constructibility of an equilateral triangle). Secondly, Euclid seems to follow Aristotle's division of first principles into particular and general principles (those first principles belonging to geometry alone) from common notions (first principles applying alike to geometry and arithmetic).

Naturally there are differences between Euclid and Aristotle, the most notable being the fact that Euclid does not cast his proofs in syllogistic form. But there is a rather close affinity between the two regarding how the axiomatic foundations are to be laid. As some of Aristotle's stresses seem to be his own, it is fairly safe to infer some connection between Aristotle's viewpoint and Euclid's practice. This tie is brought even closer when one observes that Euclid seems to adhere to Aristotle's injunction not to mix the demonstrations of one field with those of another. This attitude may have been an important factor in persuading Euclid to preserve two theories of proportion, the Eudoxean theory in Book V for ratios of magnitudes, and the Pythagorean theory of proportion in Book VII for numbers. Euclid has often been viewed as a Platonist, perhaps because the neo-Platonist Proclus classified him as such. However true that assessment might be, it seems certain that Euclid is much closer to Aristotle regarding the methodology of mathematics. But whether or not Euclid is indebted to Aristotle for some of his ideas about axiomatization, Aristotle and Euclid together form a formidable duo regarding the proper way to do mathematics. Some, indeed, including Heron and Diophantus, continued to do mathematics in a non-deductive manner, but the axiomatic approach was to become the dominant trend in Western mathematics.

GREEK DEMONSTRATIVE MATHEMATICS AND THE MEANING OF ANALYSIS

Greek mathematicians obviously did not discover their results in the polished, axiomatic way they presented them. How, then, did they find them? Some of their knowledge they inherited from others, such as the Egyptians and the Babylonians. But the Greeks went far
beyond what they learned from their predecessors, developing their own ideas and methods. A certain number of techniques which had been found to be indispensable for doing advanced geometrical work came to be known as analytic. The main works in geometrical analysis were summarized and commented upon in about 300 A.D. by Pappus in Book VII of his Collection. Near the beginning of this work we find what has come to be considered the classic description of analysis.

"Now analysis is the way from what is sought—as if it were admitted—through its concomitants in [due] order to something admitted in synthesis. For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing being sought. And we call this synthesis." (translation by Hintikka and Remes)

There are several things about this "analysis" which are worth remarking. First of all, analysis lies within the purview of axiomatic geometry. Various philosophic notions of analysis existed before and after Euclid's time, but within mathematics, the notion was confined to axiomatic mathematics, and within this stream, to deductive geometry. Analysis never seems to have been connected with Greek "algebra" or complex arithmetic, a part of mathematics which remained outside the axiomatic tradition.

In the second place, analysis is only the first stage of a two-way process. The pioneering efforts of analysis in breaking a path from the unknown to the already known must be secured and validated by proceeding in the opposite direction. Analysis is completed by synthesis.

Finally, we may note that analysis is a method of discovery. Though it is located within the axiomatic tradition and makes use of deduction, yet its purpose is not to prove known results but to uncover potential proofs of new ones. By analyzing the ramifications of the proposition stated or the problem taken, the geometer is able to ascertain whether or not it might be true or possible. Because the desired result is accepted as if it were so, i.e., is given the status of an already known result, one can bring its specifics, along with any other pertinent data that can be obtained from previously proved results, to bear upon the situation at hand, as augmented by any auxiliary constructions which may be deemed appropriate. Deductive geometrical techniques
can be applied to any part of this information, regardless of source. The commonplace nature of this procedure for a modern mathematician should not obscure its importance. Taking the sought as given and operating upon it mathematically is a powerful heuristic aid for finding plausible results and their potential proofs, even though a positive outcome is inconclusive insofar as proof is concerned.

GREEK GEOMETRICAL ANALYSIS, THEORY OF PROPORTION, AND THE NOTION OF UNIVERSAL MATHEMATICS

As far as we know, after the work of Pappus axiomatic mathematics entered a period in which nothing of any significance was done. In the middle of the fifth century, though, a commentary on the first book of Euclid's Elements was written by Proclus. At the time Proclus was the head of the neo-Platonic school of philosophy at Athens. His commentary was undoubtedly written to instruct his pupils both in Platonic doctrine and elementary mathematics. A lengthy prologue to the commentary proper provided him with ample opportunity to speculate on the nature and method of mathematics.

Proclus envisions knowledge arranged in a neo-Platonic hierarchy of sciences, the higher ones dealing with the more general forms of being. Above the special branches of mathematics, therefore, there must be a more general science which provides arithmetic and geometry with their first principles and investigates those things that are common to both. To give his reader some idea of what this universal mathematics includes, Proclus lists a number of topics. The principles which are common to all of mathematics are the Limited and the Unlimited. Theorems which are true for "all forms of mathematical knowledge" are those "theorems governing proportion,...likewise the theorems governing ratios of all kinds,...and the theorems about equality and inequality in their most general and universal aspects." Besides this, "certainly beauty and order are common to all branches of mathematics, as are the method of proceeding from things better known to things we seek to know and the reverse path from the latter to the former, the methods called analysis and synthesis."

Proclus sees analysis and synthesis, then, as methods which are common to all areas of mathematics. As such, they belong to the general science of mathematics. The theory of proportion is also a part of universal mathematics; not the only part, as Proclus is at pains to stress elsewhere, but nevertheless an important part. These ideas have a role to play in the rise of analysis in the sixteenth and seventeenth centuries, as we will shortly see.

PRELUDE TO MODERN ANALYSIS

With the decline of Greek axiomatic mathematics, the topic of mathematical analysis slipped out of sight. Various Greek philosophic
commentators tried to include mathematical analysis in their discussions of the varieties of analysis, but their remarks were usually derivative and of little value to those unfamiliar with the detailed background of Greek mathematical practice. Medieval philosophers seem to have had an entirely different concept of analysis, for they thought of analysis as a philosophic tool which was useful in discussions about the natural realm, but some went so far as to deny that mathematics could ever make use of such a method. Around the middle of the sixteenth century the topic of "method" was widely debated among Humanist philosopher-logicians and educators, and some of them stressed an idea of analysis in this context; but these discussions picked up on older philosophic notions of method and did not touch upon the mathematical tradition of analysis and synthesis. The reason why mathematical analysis was ignored all this time is quite easy to explain: all these thinkers were deprived both of the classic statement of analysis and of the classic works in analysis, not to mention mathematics in general.

With the rediscovery of ancient Greek mathematical works and the Humanists' concern to provide accurate translations based on the original sources, all this was changed. Besides the standard mathematical fare (Euclid) made available earlier in the century, Archimedes became readily accessible in 1558, Proclus in 1560, Appolonius in 1566, Diophantus in 1575, and Pappus in 1588-9. Is it any wonder, then, that mathematics was rejuvenated in the last half of the sixteenth century under this influx of first-rate mathematical works? Yet there were also some original developments which were to play a significant role in the birth of modern mathematics.

Algebra, as far as the Latin West knew, was an Arabic science. So, too, was reckoning with "Arabic" numerals. This brand of mathematics may in part be an indigenous Arabic development, but historians are fairly certain it is also the direct descendent of the Babylonian tradition mediated through Hellenistic and Hindu practical, "algebraic" mathematics. Arabic writers "demonstrate" their prescriptions by means of geometrical figures, but the character of the work is computational and algorithmic, not deductive and axiomatic.

Sixteenth century Italian mathematicians continued this algebraic tradition. By the middle of the century the solution of the cubic and quartic equations was public knowledge. The art of reckoning was also on the rise. Its use in commercial transactions, banking, and astronomy stimulated its spread. After the Bible, arithmetic texts were the most popular books published following the invention of moveable-type printing. Steviv's work in 1585 on decimal fractions showed how the Hindu-Arabic numeration system could be extended to represent fractions, and that calculating with fractions was as easy as with whole numbers.

This, then, was the general state of affairs when Viète came upon the scene with his program in analysis, first formulated in 1591 in his
Introduction to the Analytical Art. A revival of Greek mathematical learning was in full swing, and the practical, computational tradition in mathematics was asserting itself with more force than ever before. Let us see what Viète does with all this.

**THE BEGINNING OF MODERN ANALYSIS: VIETE**

Viète was a classical Humanist at heart. In his writings he strove to recover and continue the ancient Greek tradition of analysis. Yet, in actuality, he gave a fundamentally new twist to analysis. How did this come about?

Viète looked upon the venerable method of analysis as that method whereby the Greeks had discovered their results, later putting them into axiomatic form. His understanding of the essential nature of analysis was that it proceeded from the sought, as if it were given, to something already known. As we have seen, the Greeks viewed analysis as a purely geometrical method, but Viète thought he saw it at work in Diophantus' *Arithmetic* as well. For, Diophantus calculated with the unknown as if it were a determinate number, arriving in the end at a known number. Viète thought he could also detect traces of the same method in the algebra of the Italians and their Arabic predecessors, though they had defiled it with their barbarous terminology and their procedures, thus making it almost unrecognizable. It was Viète's goal, therefore, to revive the ancient method of analysis, understood algebraically, by putting it into a new and purified form, making it an analytic art.

Viète's analytic art is essentially a symbolic algebra. Prior to Viète, algebra was little more than a highly developed form of arithmetic. Though Viète himself never really broke out of the web of determinate algebraic problems to deal with indeterminate equations, yet his work, through its systematic use of an abstract symbolism, became truly algebraic. Vowels are used to represent unknown quantities and consonants to represent known quantities. Viète was thus able to treat equations in a more general way than his predecessors had, in a way which, as Descartes would note about his own procedure later, better exhibited the structure of the equation and the relations holding among the various quantities. Because of this, the fact that it was a particular solution which was being sought in any given problem became subordinate to the concern with generally applicable procedures for manipulating and solving equations. From Viète on, algebra was to become a general theory of equations. Viète's algebra still had limitations (his powers are expressed by geometrical-sounding words rather than numbers, and he interprets his arithmetic operations geometrically, only allowing quantities of like dimension or degree to appear in an equation, according to his law of homogeneity), yet the fundamental importance of his symbolic innovation cannot be denied.

In attempting to solve a mathematical problem, the analytic art starts by symbolizing all the quantities involved by letters, whether known
or unknown. The next step involves setting up the appropriate equations. Proportions, the standard way since Greek times in which relations among different quantities were exhibited, can be turned into equations by equating the product of the means with that of the extremes. This rule, according to Viète, is the most important of all the many well-known ancient "stipulations" about equations and proportions. Viète's own contribution to the construction of equations is that "supreme and everlasting law of equations or proportions, which is called the law of homogeneity...: Only homogeneous magnitudes are to be compared with one another." Once an equation has been set up, it must be manipulated according to certain "precepts" and "laws" until it is in a proper form for solving it. These rules include his instructions about how one is to reckon with "species"; i.e., with symbols that represent quantity or magnitude in general. Finally, substituting the known quantities into the equation where they belong, an appropriate procedure for resolving the equation is then performed, and the quantity being sought is actually produced. This being accomplished, one can then check to see whether the answer is indeed a solution to the problem under construction.

Against the background of all we have already said, several characteristics of Viète's analysis immediately stand out. Viète's analytic art seems to fit Proclus' notion of universal mathematics perfectly. The mathematical topics explicitly mentioned by Proclus as belonging to such a science are all present in Viète's analytic art: the theory of proportion, equalities or equations, the method of analysis. Moreover, the analytic art is applicable to both arithmetic and geometry. Viète's species can represent either numbers or geometric magnitudes because they denote objects which generalize both of them.

A closer look at the analytic art, though, reveals how far Viète is from being the reincarnation of Proclus or any other Greek mathematician. Viète's ideas are undoubtedly stimulated by Proclus' remarks on universal mathematics, yet they are also indebted to the practical, algebraic trend, probably far more than Viète wishes to admit. Everything about his analysis has a calculational cast to it. The method of analysis is misconstrued from the start in a basically algebraic manner, as we have seen. His theory of proportion makes an arithmetic criterion for proportionality apply across the board for any type of quantity whatsoever. No respectable Greek mathematician would have dreamed of doing such a thing unless the quantities involved were numbers. Viète's species, generalizing both geometric magnitude and number, have nevertheless inherited mainly the properties of number. An arithmetic theory of proportion is therefore quite understandable. Yet Viète's species are not completely like numbers, for numbers are homogeneous, all of one kind, while there are an infinite variety of species, and this must be carefully attended to in composing an equation.

Viète has wedded analysis to the practical algorithmic tradition in mathematics, which at that time was beyond the pale of respectable, axiomatic mathematics. Viète felt that his laws and procedures rested
upon the terra firma of Euclid's Elements, yet he made no effort to derive them from axioms, and it is not clear how he could have done so, for Euclid's propositions require a basic reorientation before they even begin to speak to Viète's species and equations. As for deducing the results obtained by analysis from already known results, that did not fall within the scope of the analytic art. It was observed that analytic results could be proved merely by following the argument backwards, exactly as Greek mathematicians had claimed. This very fact could make an analyst such as Viète complacent about the need for synthesis. The whole thrust of Viète's work, therefore, is different from that of Euclid.

Analysis' sole reason for existence is to solve mathematical problems, to advance mathematical knowledge. With obvious satisfaction, Viète mentions a variety of problems which his analysis is capable of solving. He then closes the Introduction to the Analytical Art on the triumphant note that "the analytical art...appropriates to itself by right the proud problem of problems, which is: TO LEAVE NO PROBLEM UNSOLVED." Viète's method of analysis is thus the true method of discovery in mathematics, "the surest finder of all things mathematical," as Viète boasts in the preface to his work.

Viète's analytic art was put to work in a program of reconstructing the works of Greek geometrical analysis excerpted in Pappus' book on analysis. This was a task undertaken by many at the time, regardless of whether they accepted Viète's view of the meaning of analysis. There were some, though, who would not rest content to restore the work of the ancients, but desired to progress beyond them. Such a man was Descartes.

THE SOURCE OF THE ANALYTIC MOVEMENT: DESCARTES

Whereas Viète had been an avowed conservative, Descartes aspired to be a revolutionary. While professing to be a staunch Catholic on religious matters, in his philosophic speculations, he elevated Reason above all else as the authoritative guide in human thought and action. Methodically doubting all things, he sought to throw over all he had learned. Previous beliefs and opinions were re-admitted in due course only if they could be founded upon a basis of clear and distinct ideas. Mathematics is for Descartes the prototype of all correct human reasoning. Only there is sure and indubitable knowledge to be found, though there is no intrinsic reason why it couldn't exist elsewhere as well, for knowledge and wisdom is all of one piece. In Descartes' Rules for the Direction of the Mind (written about 1628, though not published during his lifetime) and in his Discourse on Method (published in 1637), we can see this mathematical ideal at work. The thing that impresses Descartes most about mathematics is its certain and evident character. Though he admires deductive procedure for being able to demonstrate the truth of known mathematical results, yet his primary
concern is to fashion a method by which truth can be found. Having a right method is so important that one should not even begin the search for knowledge without it; more harm will be done than good.

On account of these opinions, Descartes asserts that the ancients must have possessed a method by which they resolved their problems, particularly in geometry. One can "recognize certain traces" of this method in the work of Pappus and Diophantus, he says, but these thinkers, along with everyone who preceded them, conspired "with a sort of low cunning" to suppress their approach, presenting their results in an ingenious axiomatic fashion instead, lest everyone would see how utterly simple it actually was to achieve what they had done.

Descartes also attributes knowledge of this method to those of his time who practiced "a certain kind of Arithmetic, called Algebra." However, in their work, it is concealed in a "vast array of numbers and inexplicable figures by which it is overwhelmed." Consequently, Descartes, just as Viète, feels compelled to rescue the real method from the obscurity in which it has lain, whether due to deception or ignorance.

Besides looking at both geometrical analysis and algebra, which he found wanting in clarity and generality, Descartes also considered logic. That, after all, was reputed to be the true method of all science. But Descartes found the logic of his time to be a mixed blessing, containing both good and bad precepts. At any rate, it was intended primarily for exposition and communication. As a tool for discovering new insights, it was barren.

Thus Descartes was thrown upon his own resources to construct a totally new method of inquiry which would incorporate the good features of all three fields while avoiding their limitations. This resulted in his invention of analysis, allegedly independently of Viète.

Descartes looks upon his mathematical method of analysis as being a particular application of his general philosophic method. Descartes' philosophic notion of analysis is described in somewhat vague terms, since he intends it to cover a wide variety of situations, but it is obviously a generalization of his understanding of mathematical analysis. In his Geometry, a work which was appended to the Discourse on Method as an illustration of the power of his general method, Descartes describes this method as it applies to geometry.

If, then, we wish to solve any problem, we first suppose the solution already effected, and give names to all the lines that seem needful for its construction—to those that are unknown as well as to those that are known. Then, making no distinction between known and unknown, we must unravel the difficulty in any way that shows most naturally the relations between these lines, until we find it possible to express a single quantity in two ways. This will constitute what we call an equation,
since the terms of one of these two expressions are equal to those of the other. And we find as many equations as there are supposed to be unknown lines; ... If there are several equations, we must use each in order, either considering it alone or comparing it with the others, so as to obtain a value for each of the unknown lines;... (translation in Struik 1969)

From this passage and others, we learn several things about Descartes' analytic method. To begin with, Descartes' notation is very close to our own. He uses letters at the beginning of the alphabet for known quantities and letters at the end for unknown quantities.

Powers of a quantity are given by numerical exponents, though the squares of quantities are usually still written as a product. Descartes operates on all these symbols without respect to type, as if they were numbers. This is true in a more strict sense for Descartes than it was for Viète, for Descartes is not bound by a law of homogeneity as Viète was. In fact, just the opposite is true. Noting that the powers of an unknown are in continued proportion (the chosen unit: $x:x^2:x^3$ etc.), something which is valid in Euclidean mathematics only if all the quantities involved are of the same type, Descartes proposes to represent all these quantities by means of a single type of quantity, by line segments. In the opening section of his Geometry, Descartes shows how arithmetical operations can be accomplished by means of geometric constructions, leading from lines to lines. This shows that magnitudes of all dimensions can be taken as homogeneous with the unit.

After one has symbolized the magnitudes involved in a problem by letters and represented them by means of line segments if necessary, one then has to determine the relations between the various known and unknown magnitudes. This results in one or more proportions, which are then translated into a series of equations. If there is enough data to determine the answers, the equations can be manipulated into a form from which the solutions can be extracted. Descartes spends a fair bit of time in his Geometry instructing the reader in the art of finding solutions when the equation is determinate. To this portion of his work belong his discussions of the quadratic formula, the factor theorem, Descartes' rule of signs, and the Fundamental Theorem of Algebra.

Descartes applies his entire apparatus to solve a particular unresolved geometrical problem mentioned by Pappus. This problem was a "locus" problem; i.e., a curve was required which would bear certain geometrical relationships to a given number of lines. Descartes' analysis proceeds by assigning letters to various line lengths, and then deriving the equation which represents the curve. Descartes has no ready-made rectangular coordinate system in which to work this problem. As is always the case, his coordinates are devised to suit the particular configuration under consideration; for the locus problem, he uses an oblique system. The important thing about his coordinates, however,
is not their geometric relationship to one another, but their algebraic one. Letting his unknowns stand for whole classes of magnitudes, an indeterminate equation results which describes how corresponding line segments are algebraically related to one another anywhere on the curve.

Let us summarize Descartes' ideas, now, putting them within their historical context. First of all, the analytic method, for Descartes as well as for Viète, was a method of universal mathematics. Universal mathematics, in Descartes' view, is that science which deals with ratios and proportions, and with order and measurement, abstractly conceived. Descartes' symbols represent magnitude in general, and so may be applied either to arithmetic or geometric problems. Descartes, too, has fallen under the spell of Proclus' vision of a universal mathematics. But while universal mathematics and the method of analysis was limited to mathematics for Viète and Proclus, in Descartes' system it becomes truly universal. Descartes explicitly proposes that his method should replace Aristotle's logic as the new "organon" for the scientific investigation of the world.

Secondly, the algorithmic, computational tradition is even more strongly at work in Descartes' view of analysis than it was in Viète's. Analysis is interpreted in terms of algebra, just as it was by Viète. Descartes does not make as much fuss about calculating with symbols, yet his symbols are more "numerical" in nature than those of Viète, for now homogeneity is present. Descartes also extends the range of influence of analysis further than Viète had done, even within mathematics proper. Viète had used analysis for both arithmetic and geometric problems, but these were essentially determinate in nature. Descartes goes beyond this to deal with indeterminate problems. In the process, analysis becomes not only a technique by which geometrical problems can be solved; it is also a method by which one can investigate the properties of geometrical figures. The fruitfulness and power of this algebraic approach is shown by Descartes' procedure for finding the normal to a curve. Geometrical analysis, fused with algebra in Descartes' own original way, becomes at last analytic geometry.

In the third place, the analytic approach was beginning to challenge the axiomatic tradition in mathematics, even making inroads on it. One followed algorithmic procedures not only in algebra and arithmetic, but, through Descartes' influence, also in geometry. Following in Descartes' footsteps, later mathematicians were to look for analytic procedures by which all the geometrical information of an equation, and hence of its curve, could be divulged. Mathematicians knew their results could be rigorously deduced, but doing so was not their style. There was too much to be learned and discovered to continually have recourse to tedious proofs. Analysis was a method of discovering new truths, not of organizing and ratifying old ones.
Descartes wrote very little on mathematics besides his Geometry. Yet he inspired many with his approach. Besides the 1637 edition, which was written in the vernacular for his French countrymen, an international (Latin) version was produced, with commentary, by van Schooten in 1649. Three years earlier van Schooten had edited Viète's works, so making them accessible to a wider public as well. Viète's influence had been felt to some extent before this time, especially in England through the works of Harriot and Oughtred. But with the publication of Descartes' work by Van Schooten, Viète's was put in the shade. It was particularly the enlarged second edition, which included the most recent research in Cartesian mathematics, published about 1660, that was to exercise such a formative influence on later seventeenth century mathematicians, including Newton and Leibniz.

The analytic movement initiated by Viète and Descartes gave mathematicians a new way to approach mathematics, first of all in arithmetic or algebra, but also in geometry. It was in geometry that the analytic method demonstrated its true mettle. Here analysis joined forces with the developments in advanced geometry that had been unleashed since the middle of the sixteenth century, when Archimedes had become available to the international community.

Archimedes was a source of inspiration to many mathematicians, who admired in particular his results on areas and volumes. However, mathematicians such as Stevin, Kepler, Cavalieri, and others disliked having to prove their results by means of the laborious Eudoxean "theory of exhaustion" with its single or double *reductio ad absurdum* argument. They preferred to use their own infinitesimal and indivisibl arguments instead. In many cases their arguments could be easily converted into a rigorous Archimedean proof. However, in the work of some seventeenth century mathematicians, the conversion is not so obvious. For in this area, too, mathematicians were more concerned with discovering new results than with proving them. By means of a variety of approaches and techniques, some of them geometrical, many results were found in that part of the calculus we now consider the province of integration.

However, this part of geometry really began to open up to mathematical treatment only with the spread of the analytic approach to mathematics. Calculating with indeterminate equations had given mathematicians a powerful tool for analyzing curves. Soon after the work of Descartes they had been able to solve tangency and extreme value problems. The problem of finding curves with a given "law of tangency" had also been treated, and some had even recognized its relation to the problem of finding areas. With the work of Newton and Leibniz, however, the "Archimedean" portion of geometry (our integral calculus) was also annexed to analysis, and the calculus was born.
It is true that the calculus owes something to the geometrical approach of a Cavalieri or a Barrow, and that its history cannot be discussed without looking at all the many particular results achieved and the various technical methods by which they were obtained. Yet the determinative factor, the thing which made the calculus what it has been since Newton and Leibniz, is analysis, analysis in the sense of Viète and Descartes. It is entirely fitting, therefore, for us to look upon the work of Newton and Leibniz in their own terms, as being the completion or crowning achievement of analysis, rather than the beginning of something brand new. With their work, the analytic movement has come of age.

THE ANALYSIS OF NEWTON: THE METHOD OF SERIES AND FLUXIONS

Sir Isaac Newton is best known for his accomplishments in "natural philosophy," for his mathematico-physical system of the world and for his work in optics. What is not so well known is that he was also an outstanding analyst. Several things besides Newton's own reputation stand in the way of our recognizing this fact.

In the first place, the two-century long competition between the analysis of Newton and the calculus of Leibniz was in the main won by Leibniz. As a result, today's viewpoint about analysis is closer to that of Leibniz. Newton's terminology and ideas appear to us to be somewhat foreign to the calculus.

Secondly, Newton's reputation as an analyst is compromised by his geometrical approach to physics. Surely this indicates his preference for Euclid and Archimedes over Descartes, whose physics, in fact, he despised because of its approach. Moreover, wasn't the geometer Barrow his teacher?

Finally, the Analytical Society at Cambridge early in the nineteenth century actively sought to replace Newtonian calculus by continental analysis. Doesn't this occur because Newton's approach was too geometric?

In response to the first difficulty, analysis must be seen against the background of Descartes' work rather than in Leibniz's terms or in the way it was viewed in the nineteenth century. Newton's analysis is different from that of Leibniz, but it is not on that account any less analytic. Nor is it less analytic because motion and geometric extension are used by Newton in thinking about his version of the derivative.

These remarks also apply to the revolt against Newton by the Analytical Society. Their desire to oust Newton in favor of Lagrange's brand of continental analysis cannot be taken as proof of any incompetence on Newton's part with respect to analysis.
The second matter is more substantial and can only be answered by looking a bit closer at what Newton says and does. Newton's geometric presentation in *Mathematical Principles of Natural Philosophy* seems to be due to his attempt to place the analysis of his day within a more traditional view of mathematics. But let's permit Newton to speak for himself on this issue.

...[I] investigated the Propositions in the Book of Principles through Analysis, and after they were investigated [I] demonstrated them through Synthesis in accordance with the law of the Ancients, who used not to admit their Propositions into Geometry before they had been demonstrated synthetically. Present-day Analysis is nothing other than Arithmetic in species. This Arithmetic can be applied to Geometrical matters, and Propositions thus found are found Arithmetically. They ought to be demonstrated synthetically in the manner of the Ancients and then finally to be regarded as Geometrical.

(from a manuscript intended as a preface to a revised edition of the *Principia*; quoted in Cohen 1971, p. 347.)

Two points may be distilled from this quote. (1) Analysis is a method of discovery, and by Newton's own admission, the method he used in finding his results. As such, however, it is inadequate for demonstrating them, which Newton feels is still necessary to doing mathematics. (2) Mathematical analysis is general arithmetic. It is possible to use it to discover geometrical results, but it is not strictly appropriate for proving them. Geometrical results must be proved, and proved geometrically. In a work that is highly geometric, therefore, such as the *Principia*, analysis is unsuitable on two counts: it is arithmetical, and it lacks demonstrative power.

It should be clear, however, that one cannot extrapolate from the approach used here to make Newton out to be an old fashioned geometer. The fact that Newton characterized his experimental method as analytic (cf. *Opticks*, Query 31) indicates how highly he valued the method of analysis. But even without leaving the *Principia* we can learn that his sympathies lie as much with analysis as with geometry.

The initial section of Book I deals with prime and ultimate ratios of quantities. The Lemmas prefaced there are given as much "to avoid the tediousness of...the method of the ancient geometers" as to provide an alternative to the "harsh" and "less geometrical" method of indivisibles. This section is, in fact, the result of Newton's attempt to put his analytic method on a rigorous basis, and so shows the imprint of analysis in its concepts if not in its techniques or logical character.

Analysis, then, had its limitations for Newton, but he nevertheless stood within the analytic tradition. His early mathematical training was strictly analytical. Newton naturally read Euclid, but the works which exercised formative influence on him were Oughtred's 1631 work on analysis, which presented Viète's specious arithmetic as the key to all of mathematics, Viète's mathematical works in van Schooten's
Newton's first mathematical treatise was written in 1669 (the ideas in it go back to 1665) and was called On Analysis by Means of Equations With An Infinite Number of Terms. The title reveals Newton's stance toward the analytic movement. Following in Wallis' steps, Newton is extending Descartes' and Viète's analysis by including infinite equations or series, thus making it an even more powerful method of discovery. Just as ordinary arithmetic has its infinite decimals for expressing certain types of quantities, so the boundaries of specious arithmetic or analysis may be enlarged to include infinite equations. Complicated algebraic expressions involving fractions or surds can be converted into an infinite series of terms by an appropriate algorithm and so made amenable to analysis. Operations and techniques which apply to finite sums can be applied to infinite sums as well, thus circumventing the need to discover new techniques for the complex quantities they represent. Whereas Cartesian analysis was restricted to algebraic curves, Newton's technique of infinite series allowed him to penetrate into the realm of "mechanical" curves.

The method of infinite series forms one part of Newton's supplement to ordinary analysis. The other half, conceived about the same time, was Newton's method of fluxions. In Newton's own words, "the methods of series and fluxions are nearly related to one another...and jointly compose one very general method of Analysis..." This joint method of analysis was first developed in Newton's work The Method of Fluxions and Infinite Series, written in 1671. Conceiving of general quantities in physical terms as being flowing quantities which increase or decrease with the passing of time, it was natural for Newton to inquire how fast they were changing at any given time. This instantaneous velocity or rate of change is the "fluxion" of the "fluent" quantity --essentially a derivative with respect to time, in our terminology.

Just as time can be thought of as being composed of moments or "Indefinitely small parts," so too can flowing quantities. Their "moments" are represented by the product of the speed or fluxion with the moment of time. If \( x \) is the fluent quantity, and \( \sigma \) the moment of time, \( \dot{x} \) will be the fluxion and \( x_0 \) the moment of \( x \). Given a particular equation relating two flowing quantities \( x \) and \( y \), one can suppose each quantity increased by its moment \( x_0 \) and \( y_0 \), substitute these new values \( x+x_0 \) and \( y+y_0 \) into the given equation, which holds for the quantities at all times, and then subtract the originally given equation. The result will be an equation which exhibits the relations between the moments \( x_0 \) and \( y_0 \). Dividing through by \( \sigma \), some terms may still contain another factor of \( \sigma \), but these "will be nothing in respect of the rest" since "\( \sigma \) is suppos'd to indefinitely little." The final result is an equation which expresses the relationship between the two fluxions \( \dot{x} \) and \( \dot{y} \).
Thus Newton gives an algorithm for proceeding from a relationship between flowing quantities to one connecting their fluxions. If the original equation involves a complex algebraic expression, the method of infinite series can be used to "simplify" it so that the same procedure can be applied. Newton also considers going in the other direction, from an equation which involves fluxions to one which shows the relationships between the associated fluents. Infinite series are used by Newton to simplify expressions in this direction as well. What we would consider to be the field of differential equations, therefore, was for Newton an integral part of his analysis, not some new branch of mathematics. Ordinary analysis solves equations which involve general quantities; Newton's analysis also solves equations that involve new quantities, fluxions of flowing quantities.

By means of his method of fluxions and infinite series, Newton is able to determine tangents to curves, whether algebraic or mechanical. But more than this, his method is able to find the areas, lengths, and volumes associated with such curves. This realm of geometry is opened up to Newton's method because he has realized in all its generality the relation between passing from a fluxion to its fluent (finding an antiderivative, in our language) and determining areas. This insight, which we rightly call the Fundamental Theorem of Calculus, gave Newton's method very great significance. Newton can now tackle those "abstruse kinds of problems" such as quadratures which, he notes, have eluded solution or have caused great difficulty to those using the ordinary method of analysis.

In this brief discussion we have only scratched the surface of Newton's mathematical thought. Yet we have said enough to see that Newton considers his work to be a continuation and enlargement of ordinary analysis. He considers his work analytic not because it deals with limiting processes, as we might be inclined to think from this side of the nineteenth century, but because it is algebraic and calculational and algorithmic in approach instead of geometric and synthetic. Leibniz held a similar opinion of his work, as we shall now see.

THE ANALYSIS OF LEIBNIZ: THE DIFFERENTIAL CALCULUS

Leibniz was a late-comer to mathematics, not starting serious study of the subject until after he was 25 years old (c. 1672). At Huygens' suggestion he read the works of Cavalieri, Pascal, and Descartes. He also studied Barrow's Geometrical Lectures and seems to have borrowed some of his ideas, though certainly not his geometrical approach. As for his debt to Newton, there isn't any. Newton and Leibniz seem to
have come to their respective positions on analysis independently of one another. Newton arrived at his ideas before Leibniz, but Leibniz was first to publish. In attracting the Bernoulli brothers to his analysis, Leibniz was also the more influential of the two. Before the seventeenth century came to a close, a text had been published on their Analysis of the Infinitely Small (1696) by L'Hôpital, a private pupil of Jean Bernoulli.

Leibniz's approach to mathematics lies squarely within the Cartesian tradition, as he himself admits. Whatever he received from other authors such as Pascal and Barrow, the general approach and the initial stimulation for his work came from Descartes. Leibniz's analysis is a continuation of ordinary Cartesian analysis into the realm of the infinitely small.

However, Leibniz also stresses the novelty of his contribution to analysis, emphasizing how far he has progressed beyond Descartes.

For when the magnitude of curved lines or the space enclosed by such is required...neither equations nor Cartesian curves can help us, and there is need of equations of a totally new kind, of constructions and new curves, and finally of a new calculus, given so far by nobody, of which if nothing else, I can now give certain examples at least, which are remarkable enough.... I have mentioned these things so that men may understand that there are certain methods in Geometry, for which they may look in vain in the work of Descartes. (Leibniz 1920, p. 187)

What is this new addition to analysis? Leibniz, calls it a "differential calculus" because it not only calculates with finite quantities, but with differences, whether finite or infinitely small. Whereas ordinary analysis calculates with certain "functions" of quantities, such as powers and roots, Leibniz's analysis also calculates with functions of variable quantities; viz, their differences or differentials.

Leibniz gives a number of rules for determining the differences of quantities. We may take the differential of a product as an example. Leibniz claims that \( dxy = xdy + ydx \). His original proof of this is by means of an infinitesimal argument, which goes like this.

\[
dxy = (x+dx)(y+dy) - xy = xdy + ydx + dxdy
\]

the omission of the quantity \( dxdy \), which is infinitely small in comparison with the rest, for it is supposed that \( dx \) and \( dy \) are infinitely small... will leave \( xdy + ydx; \ldots \) (Leibniz 1920, p. 143)

Leibniz later tried to avoid such infinitesimal arguments through the use of his continuity postulate, as we shall see.
Once one knows how to find differentials of complex quantities, including fractional and irrational quantities, these algorithms can be used to transform any equation directly into its associated "differential equation," thus bypassing the lengthy detour which was usually needed in the Cartesian tradition in order to first put the equation into a manageable form. Following this, one can use the differential equation to determine tangents, extreme values, convexity, points of inflection, etc. Moreover, "transcendent curves," those curves which Descartes called "mechanical" and which his methods were unable to touch, are susceptible to Leibniz's differential calculus. This alone shows the superiority of Leibniz's analysis over that of Descartes.

Yet there is more, just as there was for Newton. Leibniz's differential calculus is also applicable to quadrature problems. Leibniz very early recognized that "differences and sums are the inverses of one another," as he says. Since an area can be thought of as an infinite series of lines or infinitely thin rectangles, "the general problem of quadratures can be reduced to the finding of a line [i.e., a curve] that has a given law of tangency..." The differential calculus, used in reverse, thus holds the key to this part of geometry, too.

The power of Leibniz's variety of analysis, therefore, resides in its calculation of differentials. It is this which makes his calculus transcend ordinary analysis. But it remains analysis nevertheless. Once having introduced the differential functions into analysis and shown how to compute them, Leibniz treats them much the same as other quantities. Equations are constructed and solved by algebraically manipulating knowns and unknowns, including differentials, and curves are analyzed by means of them.

It is not surprising, however, that Leibniz calls his analysis differential calculus. The computational and algorithmic aspects of analysis are always in the foreground. This is true to such an extent that Leibniz stresses formal symbolic procedures over against the use of geometric figures. It is in this setting that we must place Leibniz's invention of his famous symbolism for calculus (∫ for sum, dX for the differential of X). As he reflected back on the origin of his calculus some years later, Leibniz remarks that his notation was consciously designed so that "the imagination [would be] freed from a perpetual reference to diagrams." This formalistic, operational outlook seems to originate with Leibniz, though it only accents a tendency which had been present in analysis ever since its modern formulation by Viète. Eighteenth century continental mathematicians were to ingest this bias of Leibniz along with adopting the technical achievements of the differential calculus. When Lagrange says in 1788 that his version of mechanics converts it into a branch of analysis, he means that it is chock full of mechanical procedures, algebraic operations, and equations, and that it is devoid of geometric diagrams.
Leibniz's preference for analysis over geometry also shows up in his attitude toward proof and logical rigour. In defending his differential calculus against an attack by the Dutch mathematician Nieuwentijdt, Leibniz says that his method of analysis is merely a method of discovery. Its justification lies in its fruitfulness, in its ability to resolve intricate problems with ease and extend the domain of known results. However, Leibniz held that the results gained by his method could be proved rigorously in the standard Archimedean fashion by whoever cared to do so. He also indicated that a demonstration of his infinitesimal method might be based upon his postulate of continuity. This is the way Leibniz formulated this principle:

"In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included." (Leibniz 1920, p. 147)

Translated into the context of his differential calculus, this principle means that if something holds good when a quantity is deemed indefinitely small, then it must also be true when that quantity is considered to be nothing at all. Quite a convenient principle: whatever is true up to the limit must also be true at the limit. As far as I know, no one took up Leibniz's hint and tried to base the differential calculus upon this philosophic continuity postulate, though it was asserted as a true principle in the form I just gave it during the late eighteenth and early to mid nineteenth centuries. In the main, Leibniz's followers were too busy broadening out analysis by applying it to physical problems to bother with logical niceties like a proper conceptual foundation for their methods. Archimedes was always there to fall back upon when someone requested a rigorous demonstration.

ANALYSIS IN THE WAKE OF NEWTON AND LEIBNIZ: ONWARD AND FORWARD

If we compare Newton's method of series and fluxions with Leibniz's differential calculus, the following picture emerges. First of all, both Newton and Leibniz stand within the new orthodoxy of the analytic movement. Analysis is for them, as it was for Descartes, the method by which advanced geometrical research is done.

Secondly, Newton and Leibniz further extend and advance analysis by developing new analytic techniques. Existing analytic procedures are simplified and their range of application is extended to more complex curves. Both recognize the property we now call the Fundamental Theorem of Calculus, and so are able to bring quadratures within the scope of the analytic method.

Naturally, there are also differences between Newton and Leibniz. Some of these are real and important differences, but they are still differences within a common heritage, that of analysis. We will mention two differences which had historical consequences for the further development of analysis.
First of all, Newton and Leibniz have different bases for their new techniques. Newton accepts infinite series expansions by analogy with arithmetic. Fluxions are invented by importing a physical concept into geometry. Leibniz, on the other hand, uses infinitely small quantities to obtain his results, drawing more upon seventeenth century developments in geometry. However, as his calculus matures, he seems to repudiate this foundation, treating infinitesimals as ideal elements which are to be used merely to facilitate computation. Thus, his calculus appears in the end to be a body of formal algorithmic procedures without a proper mathematical basis.

The second matter is closely related to the first. Both Newton and Leibniz view analysis as a powerful method for discovering new results, but they differ regarding the need for proof and logical foundation. Ironic as it may seem, Newton the physicist is concerned to give his techniques and ideas a proper synthetic foundation, while Leibniz the logician thinks it is unnecessary.

These differences color the development of analysis throughout the eighteenth century. British mathematicians maintained a more geometric and physical view of calculus and were more sensitive to foundational critique, such as that of Bishop Berkeley. Continental mathematicians were more formalistic in their approach and tended to ignore the issue of foundations. It was not so much that they were undecided about the basis of their field, as that they were unconcerned about it. On the whole they were too busy pushing back the frontiers in the various areas of physical science to regress to the style of the old school in mathematics.

If we had time (it would take at least another talk), we could go on to discuss the foundational developments in analysis which start as a trickle in the last half of the eighteenth century with the attention paid to the "metaphysics of the calculus" by D'Alembert and Lagrange, among others, then widen considerably through Cauchy's efforts to make analysis as rigorous as geometry, and finally come into their own with the arithmetization program of Weierstrass, Cantor, and Dedekind. These developments reveal a more deductive consciousness among mathematicians, while at the same time showing their increasing attachment to the arithmetical character of analysis. Notwithstanding this trend toward rigor, the axiomatic method did not emerge in the field of analysis.

THE MODERN AXIOMATIC APPROACH TO MATHEMATICS:

PASCH, PEANO, AND HILBERT

A rigorous axiomatic approach to mathematics first surfaced in the very field it had been all along, in geometry. The work which was historically significant in this respect was Moritz Pasch's deductive treatise on projective geometry published in 1882. Pasch requires
that his system of axioms be the logical foundation for geometry in a strict or complete sense. No idea can be used in an argument unless it is explicitly grounded in the axioms. To proclude smuggling anything foreign into the superstructure, one must rely neither upon the geometric figures, nor upon any assumed meaning or interrelation of the concepts used—unless, of course, it has been asserted by, or can be shown to follow from, the axioms.

Pasch's ideal, therefore, is to make geometry explicitly and totally axiomatic in order to avoid deductive gaps in proofs. He is not a formalist, however, and he admits that meaning may be valuable in constructing an argument. But while meaningful ideas may guide an argument, it should not be necessary for them to do so. If, on the contrary, meaning must be imported in order to make a proof go through, the argument must be judged inadequate to establish the proposition under consideration.

After Pasch, various Italian mathematicians, including Peano, continued this axiomatic trend, both in and out of geometry. The work which really established the axiomatic method in mathematics, however, was Hilbert's Grundlagen der Geometrie, first published in 1899. By axiomatizing a field that was familiar to all mathematicians, regardless of capability or interests, Hilbert showed the mathematical community the investigative potential of the axiomatic method. For in this work, the axiomatic approach comprises far more than positing a number of axioms and stringently deducing theorems from them. It also involves investigating the logical relationships which hold among the axioms.

In order to do this, Hilbert found it necessary to go beyond Pasch and adopt a formalist position on mathematics. The primitive terms must remain undefined and should not be thought of as having any definite meaning. Their "meaning," if it can be called that, is implicitly defined by the axioms which mention them and stipulate their interrelations. Theorems are to be strictly deduced from the axioms, and so will be true for any valid interpretation of the given terms, regardless of whether it is the usual, or intended, interpretation.

Hilbert was able to establish the consistency of certain sets of axioms and the independence of various axioms from others by constructing a variety of models. In Hilbert's hands the axiomatic method became a tool for systematically pursuing foundational research in mathematics, and not merely a means of logically organizing an already developed field.

The axiomatic method was also taken up by mathematicians who had little interest in logic or foundations. Because it was congenial both to general theories, such as group theory, and to more specific theories, such as that of the continuum, its use spread in mathematics. The result has been that which we mentioned at the start of the talk—an almost universal acceptance of the axiomatic method. Hilbert
forecast this in 1918 with the following words:
"Everything that can be the object of mathematical
thinking, as soon as the erection of a theory is
ripe, falls into the axiomatic method and thereby
directly into mathematics. By pressing to ever
deeper layers of axioms...we can obtain deeper
insights into scientific thinking and learn the
unity of our knowledge. Especially by virtue of
the axiomatic method mathematics appears called
upon to play a leading role in all knowledge."
Whether we like it or not, the formal, axiomatic approach intro-
duced by Hilbert has become a way of life in mathematics.

Acknowledgements

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an earlier draft of this paper.

FOOTNOTES

1 I am indebted to Stillman Drake for suggesting this idea. Charles
Jones explores the relation between Euclid's two theories of propor-
tion and the possible connection between Aristotle and Euclid gener-
ally in the first part of his thesis on Stevin.

2 This viewpoint was first put forward, as far as I know, by Michael
Mahoney, who sees the analytic movement of the sixteenth and seven-
teenth centuries as a "Kuhnian revolution" in the history of mathe-
matics. Jacob Klein earlier asserted that modern mathematics was
born with the work of Viète, but his viewpoint is somewhat different,
and he does not carry his analysis to the end of the seventeenth
century. Thomas Hawkins gave a talk at the Canadian Learned Society
Conference in June of 1978, exploring Mahoney's thesis, and this
helped confirm its likelihood in my mind, especially with respect
to Newton.
Bibliography


