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On the Fattening of Lines in P3

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On the Fattening of Lines in P^3

Abstract

We follow the lead of Bocci and Chiantini and show how differences in the invariant α can be used to classify certain classes of subschemes of P^3 . Specifically, we will seek to classify arithmetically Cohen-Macaulay codimension 2 subschemes of P^3 in the manner Bocci and Chiantini classified points in P^2 . The first section will seek to motivate our consideration of the invariant α by relating it to the Hilbert function and γ , following the work of Bocci and Chiantini, and Dumnicki, et. al. The second section will contain our results classifying arithmetically Cohen-Macaulay codimension 2 subschemes of P^3 . This work is adapted from the author's Ph.D. dissertation.

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On the fattening of lines in \mathbf{P}^3 [☆]

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ABSTRACT

We follow the lead of [2] and show how differences in the invariant α can be used to classify certain classes of subschemes of \mathbf{P}^3 . Specifically, we will seek to classify arithmetically Cohen–Macaulay codimension 2 subschemes of \mathbf{P}^3 in the manner Bocci and Chiantini classified points in \mathbf{P}^2 . The first section will seek to motivate our consideration of the invariant α by relating it to the Hilbert function and γ , following the work of [2,5]. The second section will contain our results classifying arithmetically Cohen–Macaulay codimension 2 subschemes of \mathbf{P}^3 . This work is adapted from the author’s Ph.D. dissertation [11].

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1. The importance of α

Let k be an algebraically closed field of arbitrary characteristic. Much is known about finite sets of reduced points $Z \subseteq \mathbf{P}^2$ over k . In particular, [7] classified all possible Hilbert functions of finite sets of reduced points in \mathbf{P}^N over k . However, not much is known about the double scheme, $2Z$ (but see [8,6]).

Definition 1.1. In general, the m -th symbolic power of a homogeneous ideal $I \subseteq R = k[\mathbf{P}^N]$ is

$$I^{(m)} = R \cap \left(\bigcap_{P \in \text{Ass}(I)} (I^m R_P) \right),$$

where $\text{Ass}(I)$ denotes the set of associated primes of I and R_P is the ring R localized at the prime P .

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When I is the ideal of a complete intersection, $I^{(m)} = I^m$ (see [13, Lemma 5, Appendix 6]), and thus if $I = (L_1, L_2)$ (where L_1, L_2 are linear forms) is the ideal of a linear codimension 2 complete intersection (e.g., a point in \mathbf{P}^2 , or a line in \mathbf{P}^3), we have $I^{(m)} = (L_1, L_2)^m$.

Definition 1.2. Let $Z \subseteq \mathbf{P}^N$ be a reduced subscheme defined by $I = I(Z)$. The double scheme (often called the double point scheme if Z is a set of reduced points, or the fattening) is the subscheme of \mathbf{P}^N defined by $I^{(2)}$ and denoted $2Z$.

In this paper, we follow the lead of Bocci and Chiantini [2] and others in studying the number $\alpha(I)$, where I is the ideal of a reduced codimension 2 subscheme. Recall that, if I is a nonzero homogeneous ideal in $k[\mathbf{P}^N]$, the number $\alpha(I)$ is the degree of a nonzero polynomial of least degree in I . (Equivalently, if I_d denotes the homogeneous component of I of degree d , $\alpha(I) = \min \{d : I_d \neq 0\}$.)

Now, it is not difficult to see that the number α is the degree in which the Hilbert function of the quotient R/I first deviates from that of the ring $R = k[\mathbf{P}^N]$. Indeed, recall that the Hilbert function of the quotient of a homogeneous ideal $I \subseteq R =$ in degree t is $H(R/I, t) = \dim_k(R_t) - \dim_k(I_t)$. If $t < \alpha(I)$, $\dim_k(I_t) = 0$, hence $H(R/I, t) = \dim_k(R_t) = \binom{t+N}{N}$.

Thus, Bocci and Chiantini, rather than compute α (or even Hilbert functions) of various planar point configurations in \mathbf{P}^2 or their symbolic powers, chose to study the difference $t := \alpha(2Z) - \alpha(Z)$. This is related to the Waldschmidt constant

$$\gamma(I) := \lim_{m \rightarrow \infty} \frac{\alpha(I^{(m)})}{m}.$$

Understanding $\alpha(I^{(m)})$ for all $m \geq 1$ is a difficult task, while classifying Z based on the difference t is more tractable (though, as t increases, it grows more difficult).

An important first observation about t is that $t \geq 1$ always holds; indeed, if k has characteristic 0, let F be a form of minimal degree $\alpha(2Z)$ vanishing to order at least 2 at each point of Z . Then the partial derivatives of F vanish on Z , and the degree of the partial derivatives is less than the degree of F . If k has characteristic $p > 0$, then it may happen that every partial derivative of F is identically 0. In that case, F is the p -th power of some form G , which vanishes at each point of Z , and thus $t \geq 1$.

We follow the lead of [2] and say that a subscheme $Z \subseteq \mathbf{P}^N$ has type $(d - t, d)$ if $\alpha(Z) = d - t$ and $\alpha(2Z) = d$.

Bocci and Chiantini examine cases when t is small; specifically, they consider $t = 1, 2$. When $t = 1$, they use Bézout's Theorem to find:

Theorem 1.3. (See Example 3.1, Proposition 3.2, and Theorem 3.3 of [2].) *Let $Z \subseteq \mathbf{P}^2$ be a finite set of points. Then $t = 1$ if and only if either Z is a set of collinear points and $\alpha(Z) = 1$ or Z is a star configuration of $\binom{d}{2}$ points and $\alpha(Z) = d - 1$.*

That $\alpha(Z) = 1$ and $\alpha(2Z) = 2$ when Z is a set of collinear points is clear. A star configuration of points in \mathbf{P}^2 is the finite subset Z of $\binom{d}{2}$ points of pairwise intersection of d lines, where $d \geq 3$. See Fig. 1 for a star configuration Z when $d = 5$. Let F be the product of the five linear forms corresponding to the lines, and G be the form F divided by one of the linear forms. Then it is clear that F vanishes to order 2 at each of the 10 points and G vanishes to order at least 1 at each point; it is known that F and G are forms of minimal degree vanishing to order 2 and 1, respectively, which means that Z has type $(4, 5)$.

When $t = 2$, Bocci and Chiantini also obtain classification results, though these are much more complicated. The situation can be roughly described as follows: either $\alpha(2Z) = 4$ and Z lies in a conic, or $\alpha(2Z) > 4$ and Z lies in the nodes of the union of rational curves.

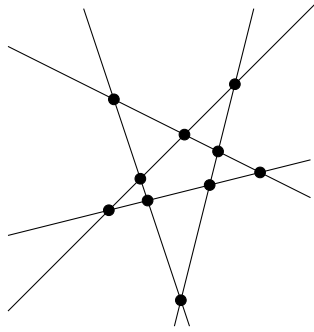


Fig. 1. A star configuration formed by the pairwise intersection of 5 lines in \mathbf{P}^2 .

There are several possible avenues for generalizing these results; the first we consider is to look at higher symbolic powers. We borrow the following notation from [5]:

Notation 1.4. Let $Z \subseteq \mathbf{P}^2$ be a finite fixed set of arbitrary points. Then we use the notation $\alpha_{m,n}(Z) := \alpha(I^{(m)}) - \alpha(I^{(n)})$ for $m > n$.

In [5], Dumnicki et al. obtain stronger results by requiring the successive differences $\alpha_{m+1,m}$ to be constant as m increases.

They then prove:

Theorem 1.5. (See Theorems 3.1 and 4.14 of [5].) If

$$\alpha_{2,1}(Z) = \alpha_{3,2}(Z) = \dots = \alpha_{t+1,t}(Z) = d,$$

then

- (1) for $d = 1$ and $t \geq 2$ the set Z is contained in a line, i.e., $\alpha(Z) = 1$;
- (2) for $d = 2$ and $t \geq 4$ the set Z is contained in a conic, i.e., $\alpha(Z) = 2$.

Moreover, both results are sharp, i.e., there are examples showing that one cannot relax the assumptions on t .

The authors believe that such a result should be true for cubics as well.

Another recently-explored avenue is to points in $\mathbf{P}^1 \times \mathbf{P}^1$; in [1], the authors extend the results of [2,5] to bi-homogeneous ideals over $\mathbf{P}^1 \times \mathbf{P}^1$.

A third avenue for generalizing the results of Bocci and Chiantini is to consider subschemes of higher dimensional projective spaces, and this is the direction we will take in the remainder of this note. However, rather than look at point configurations, we will examine configurations of lines in \mathbf{P}^3 . Although the study of fat lines has been the subject of some recent study (see e.g., [9,10,4]), little is still so far known about them.

Herein, we begin the task of extending the $t = 1$ result of Bocci and Chiantini to lines in \mathbf{P}^3 by examining line configurations which are arithmetically Cohen–Macaulay (ACM). At the time of this writing, all known examples of line configurations with $t = 1$ are ACM. Indeed, question of the existence of $t = 1$ line configurations which are not ACM is open at this time. One of our goals in this work is to draw further attention to this situation by examining the ACM case. We also show that the usual notion of a star configuration (as defined in [2] and generalized in [6], among others) is not general enough, and introduce the notion of a pseudo-star configuration. This work is based on [11].

2. Lines in \mathbf{P}^3

Throughout the remainder, let $S = k[\mathbf{P}^3] = k[x, y, z, w]$ and $R = k[\mathbf{P}^2] = k[x, y, z]$ be the homogeneous coordinate rings of \mathbf{P}^3 and \mathbf{P}^2 , respectively.

Broadly speaking, the two types of configurations of lines in \mathbf{P}^3 we will discuss are the coplanar configurations and the pseudo-star configurations.

Definition 2.1. A pseudo-star configuration (or pseudostar) of lines in \mathbf{P}^3 is a finite collection of lines formed by the pairwise intersection of hyperplanes such that no three of the hyperplanes meet in a line.

There is a growing body of literature on the study of star configurations (see [6] and the references therein). Indeed, star configurations were one of the first examples studied in [3] in which the resurgence $\rho(I)$ was introduced. The easiest examples, of course, are star configurations of points in \mathbf{P}^2 , but star configurations can be defined in any codimension in any projective space.

As defined in [6], a star configuration of lines in \mathbf{P}^3 is a collection of lines formed by the pairwise intersections of hyperplanes which meet properly, meaning that the intersection of any j of the hyperplanes is empty or has codimension j . For the case of the pseudostars, we replace the requirement that the planes meet properly with the requirement that no three of the planes meet in a line; therefore, it may be that in a pseudostar in \mathbf{P}^3 , more than three planes meet in a single point.

The easiest example of a pseudostar in \mathbf{P}^3 is a star configuration of lines.

Another easy example of a pseudostar in \mathbf{P}^3 is a projective cone over a star configuration of points in \mathbf{P}^2 :

Example 2.2. Suppose $I \subseteq R$ defines a star configuration Z of points in \mathbf{P}^2 . The projective cone over Z is a subscheme of \mathbf{P}^3 defined by the extension IS of I to S . This is an example of a pseudostar.

The proof of our main theorem will be powered by the notion of arithmetically Cohen–Macaulay subschemes. In particular, for an ACM subscheme $X \subseteq \mathbf{P}^N$ we will exploit the relationship between $\alpha(X)$ and $\alpha(X \cap H)$, where H is a general hyperplane.

Definition 2.3. A subscheme $X \subseteq \mathbf{P}^N$ is arithmetically Cohen–Macaulay (ACM) if the homogeneous coordinate ring $k[\mathbf{P}^N]/I(X)$ of the subscheme is Cohen–Macaulay.

Several familiar linear configurations are ACM.

Lemma 2.4. Any collection of coplanar lines in \mathbf{P}^3 is ACM.

Proof. If $I \subseteq S$ is the ideal of coplanar lines, then I is a complete intersection ideal, and thus S/I is Cohen–Macaulay. \square

Lemma 2.5. Let \mathbb{L} denote a finite union of lines in \mathbf{P}^3 . If \mathbb{L} is a star configuration of lines in \mathbf{P}^3 or a projective cone over a star configuration of points in \mathbf{P}^2 , \mathbb{L} and $2\mathbb{L}$ are ACM.

Proof. If \mathbb{L} is a star configuration of lines in \mathbf{P}^3 , then \mathbb{L} and $2\mathbb{L}$ are ACM by [6, Proposition 2.9 and Theorem 3.1], respectively. Suppose \mathbb{L} is a projective cone over a star configuration Z in \mathbf{P}^2 . Then $I(\mathbb{L}) = I(Z)S$, and $(R/I(Z))[w] \cong S/I(Z)S = S/I(\mathbb{L})$. Since $R/I(Z)$ is Cohen–Macaulay, so is $(R/I(Z))[w]$, and hence also $S/I(\mathbb{L})$. Therefore \mathbb{L} is ACM. A similar argument can be carried out for $I(2Z) = (I(Z))^{(2)}$. \square

Proposition 2.6. Pseudostars and their symbolic squares are ACM.

Proof. The reduced case was proved, though not explicitly, in [6, Proposition 2.9] (but see [6, Remark 2.13]). The symbolic square case can be found in the first part of the proof of [6, Theorem 3.2], as the assumption that the hyperplanes meet properly can be relaxed to the assumption that no three hyperplanes contain a line. \square

Proposition 2.7. (See Corollary 1.3.8 of [12].) Let $X \subseteq \mathbf{P}^N$ be an arithmetically Cohen–Macaulay scheme of dimension at least 1, and suppose $H \subseteq \mathbf{P}^N$ is a general hyperplane. Let $X \cap H$ denote the general hyperplane section of X , $S = k[\mathbf{P}^N]$, and $R = S/I(H) \cong k[\mathbf{P}^{N-1}]$. Then the Hilbert function of $R/I(X \cap H)$ is given by

$$H(R/I(X \cap H), t) = H(S/I(X), t) - H(S/I(X), t - 1).$$

A useful corollary of Proposition 2.7 is the following.

Corollary 2.8. Suppose $X \subseteq \mathbf{P}^N$ is an arithmetically Cohen–Macaulay scheme of dimension at least 1, and $H \subseteq \mathbf{P}^N$ is a general hyperplane. If $X \cap H$ denotes the general hyperplane section of X , then $\alpha(X) = \alpha(X \cap H)$.

Proof. This follows immediately from Proposition 2.7 and the definitions of the Hilbert function and α . \square

Corollary 2.9. Let \mathbb{L} be a pseudostar in \mathbf{P}^3 formed by the pairwise intersection of d planes, no three of which contain any line. Then $\alpha(\mathbb{L}) = d - 1$ and $\alpha(2\mathbb{L}) = d$.

Proof. We first fix our notation. Let $H_1, H_2, \dots, H_d \subset \mathbf{P}^3$, $d > 2$ (if $d \leq 2$, the lines resulting from the pairwise intersection of the hyperplanes will be coplanar) be hyperplanes, no three of which contain any line. Set $\ell_{ij} = H_i \cap H_j$ for all $i < j$, and put $\mathbb{L} = \bigcup_{1 \leq i < j \leq d} \ell_{ij}$. Then \mathbb{L} is a pseudostar.

We first show that $\alpha(\mathbb{L}) = d - 1$. By Corollary 2.8, it is enough to show that the general hyperplane sections of \mathbb{L} form a star configuration of points in \mathbf{P}^2 .

A general hyperplane H meets each H_i in a line L_i ; as H is general, L_i meets each ℓ_{ij} , $j \neq i$ in distinct points $p_{ij} \in H \cong \mathbf{P}^2$. The points p_{ij} , $j \neq i$, form a star configuration of points in $H \cong \mathbf{P}^2$, as each line L_i contains $d - 1$ points p_{ij} , $j \neq i$; each point p_{ij} lies on exactly two lines, L_i and L_j , hence we have exactly $\binom{d}{2}$ points. Thus, the general hyperplane sections of \mathbb{L} form a star configuration.

To see that $\alpha(2\mathbb{L}) = d$, note that $d \geq \alpha(2\mathbb{L}) > \alpha(\mathbb{L}) = d - 1$. \square

The following proposition shows that if a general hyperplane intersects three or more lines in \mathbf{P}^3 in collinear points, the lines must lie in a plane.

Proposition 2.10. A general hyperplane intersects $d \geq 3$ non-coplanar lines in \mathbf{P}^3 in d non-collinear points.

Proof. Note that it is sufficient to prove the proposition for $d = 3$, and let $\mathbb{L} = \ell_1 \cup \ell_2 \cup \ell_3$ be the union of 3 non-coplanar lines ℓ_1, ℓ_2, ℓ_3 . Fix a general point $p \in \ell_1$, and let π be the projection from p . Observe that, if $\pi(\ell_2) = \pi(\ell_3)$, then ℓ_2 and ℓ_3 span a plane meeting ℓ_1 at p , which is impossible since p is general, unless \mathbb{L} is coplanar. Thus, $\pi(\ell_2) \neq \pi(\ell_3)$.

Then take general points $q_2 \in \pi(\ell_2) \setminus \pi(\ell_3)$ and $q_3 \in \pi(\ell_3) \setminus \pi(\ell_2)$. Their preimages $p_2 \in \ell_2$, $p_3 \in \ell_3$ determine, with $p \in \ell_1$, a general plane meeting the three non-coplanar lines in non-collinear points. \square

Another way to say this is:

Corollary 2.11. If $d \geq 3$ lines in \mathbf{P}^3 intersect a general hyperplane H in collinear points, then the lines are coplanar.

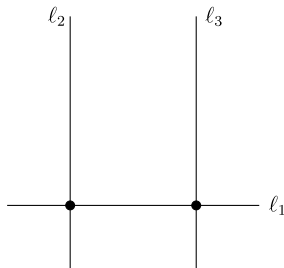


Fig. 2. Three lines ℓ_1, ℓ_2, ℓ_3 in \mathbf{P}^3 such that $\ell_2 \cap \ell_3 = \emptyset$.

We set the following notation.

Notation 2.12. Let $H_1, H_2, \dots, H_d \subset \mathbf{P}^3$ be hyperplanes, no three of which contain any line. Set $\ell_{ij} = H_i \cap H_j$ for all $i < j$, and put $\mathbb{L} = \bigcup_{1 \leq i < j \leq d} \ell_{ij}$.

We now come to the main result of this work, which describes an extension of Bocci and Chiantini's $t = 1$ result for points in \mathbf{P}^2 . In \mathbf{P}^2 every codimension 2 subscheme is ACM, as all finite sets of points in any \mathbf{P}^N are ACM. In higher dimensions, not every codimension 2 subscheme in \mathbf{P}^N is ACM (e.g., three skew lines in \mathbf{P}^3). However, the natural generalization of the Bocci–Chiantini result seems to be for ACM codimension 2 subschemes (but see [Question 3.1](#)).

Theorem 2.13. Let \mathbb{L} be a union of lines $\ell_1, \ell_2, \dots, \ell_s$.

- (a) If \mathbb{L} is ACM of type $(d - 1, d)$ for some $d > 1$, then \mathbb{L} is either a pseudostar or coplanar.
 (b) If \mathbb{L} is either a pseudostar or coplanar, then \mathbb{L} has type $(d - 1, d)$ for some $d > 1$.

Proof. We begin by proving (a), and first treat the cases in which $1 \leq s \leq 3$ in an ad hoc fashion.

Indeed, if $s = 1$, we have a single line, which is coplanar, so (a) holds.

For $s = 2$, either the lines meet, in which case they are coplanar, or the lines are skew. If the lines ℓ_1, ℓ_2 are skew, then, without loss of generality, we may take $I(\ell_1) = (x, y)$ and $I(\ell_2) = (z, w)$, so $I(\mathbb{L}) = (x, y) \cap (z, w)$, $\alpha(\mathbb{L}) = 2$, and $\alpha(2\mathbb{L}) = \alpha((x, y)^2 \cap (z, w)^2) = 4$ so \mathbb{L} has type $(d - 2, d)$. In either case, if $s = 2$, (a) holds.

If $s = 3$, we have three possible configurations. If the lines meet in a single point, they are either coplanar or a pseudostar. If the lines do not meet in a single point but intersect pairwise, they are coplanar. The last case involves lines ℓ_1, ℓ_2, ℓ_3 such that ℓ_2 and ℓ_3 do not meet, but $\ell_2 \cap \ell_1 \neq \emptyset$ and $\ell_3 \cap \ell_1 \neq \emptyset$, as in [Fig. 2](#). In this case, we can, after an appropriate change of coordinates, assume $I(\ell_1) = (x, z)$, $I(\ell_2) = (y, z)$, and $I(\ell_3) = (x, w)$. One can easily verify using a computer algebra system that $\mathbb{L} = \ell_1 \cup \ell_2 \cup \ell_3$ is of type $(2, 4)$ and is not a pseudostar (in particular, a pseudostar of three lines meets in a point). Thus, (a) is satisfied for $1 \leq s \leq 3$. To finish the proof we consider the case that $s \geq 4$.

Suppose \mathbb{L} has type $(d - 1, d)$ for some $d \geq 2$, and let H denote a general hyperplane. As \mathbb{L} is ACM, we can apply [Proposition 2.7](#) to \mathbb{L} to see that $\alpha(\mathbb{L}) = \alpha(\mathbb{L} \cap H) = d - 1$, and since $d = \alpha(2\mathbb{L}) \geq \alpha(2(\mathbb{L} \cap H)) > \alpha(\mathbb{L} \cap H) = d - 1$ (see [\[2\]](#)), the general hyperplane sections $\mathbb{L} \cap H$ must have type $(d - 1, d)$ in $H \cong \mathbf{P}^2$. By [\[2\]](#), this means that the general hyperplane sections $\mathbb{L} \cap H$ of \mathbb{L} are either a set of collinear points or a star of points in \mathbf{P}^2 .

If $\mathbb{L} \cap H$ is a set of collinear points, we must have that \mathbb{L} is a set of coplanar lines (see [Proposition 2.10](#)). Otherwise, by [Proposition 2.10](#) (since $s \geq 4$ and thus $s - 1 \geq 3$) we have d (non-disjoint) collections of $d - 1$ collinear points (in fact, we have $\binom{d}{2}$ points total, since $\mathbb{L} \cap H$ is a star in $H \cong \mathbf{P}^2$). Each of the $\binom{d}{2}$ points is the hyperplane section of exactly one of the ℓ_{ij} 's, so we must have $s = \binom{d}{2}$ lines ℓ_{ij} , with d (non-disjoint) collections of $d - 1 \geq 3$ coplanar lines. Moreover, since we have d hyperplanes meeting in $\binom{d}{2}$ lines, it must

be that no three hyperplanes meet in a line, or else we would have strictly fewer than $\binom{d}{2}$ lines, and thus strictly fewer than $\binom{d}{2}$ hyperplane sections. Thus, \mathbb{L} forms a pseudostar.

We now turn to (b). Note that if \mathbb{L} is coplanar, \mathbb{L} clearly has type $(1, 2)$. If \mathbb{L} is a pseudostar, then by Corollary 2.9 \mathbb{L} has type $(d - 1, d)$ for some $d > 1$. \square

3. Future work

It seems as that pairing this approach with an inductive argument may generalize Theorem 2.13 to ACM codimension 2 subschemes of \mathbf{P}^N , $N > 3$, but this has not yet been explored.

There are several other avenues for future work.

We made use of the assumption that the lines in question in \mathbf{P}^3 are arithmetically Cohen–Macaulay (ACM). Our hope is that this future work will result in the relaxation (or removal) of this assumption. To that end, a natural question is:

Question 3.1. Does there exist a configuration of lines of type $(d - 1, d)$ which is not ACM?

Indeed, computational examples suggest that adding a disjoint line to a $(d - 1, d)$ (ACM) configuration yields a non-ACM configuration of type $(d, d + 2)$.

If the answer to Question 3.1 is no, then the ACM hypothesis in Theorem 2.13 is unnecessary.

Another interesting question (suggested by Juan Migliore) is:

Question 3.2. Which reduced (possibly irreducible) curves in \mathbf{P}^3 have type $(d - 1, d)$ for some $d > 1$?

As every finite set of points in \mathbf{P}^N (for any $N \geq 1$) is ACM, another natural question to ask is:

Question 3.3. Which configurations of points in \mathbf{P}^3 have type $(d - 1, d)$?

In [2], the authors also classify configurations of points in \mathbf{P}^2 which have type $(d - 2, d)$. Thus, we ask:

Question 3.4. Which arrangements of lines in \mathbf{P}^3 have type $(d - 2, d)$? Which arrangements of points in \mathbf{P}^3 have type $(d - 2, d)$?

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