5-29-2015

Symbolic Powers of Edge Ideals

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Keywords
commutative algebra, ideals (algebra), graph theory

Disciplines
Algebraic Geometry

Comments
Presentation at the 20th Biennial Conference of the Association for Christians in the Mathematical Sciences held at Redeemer College in Ancaster, Ontario, Canada on work that sprung out of a Kuyper Scholars Program project in Spring 2015 connecting algebra and graph theory.

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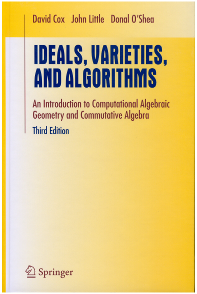
Our project

Background: a student approached me to do an honors contract in a special topics course.



My research area: commutative algebra/algebraic geometry

Our situation

Let k be an algebraically closed field (e.g., $k = \mathbb{C}$).

We will primarily consider homogeneous ideals $I \subseteq R = k[x_0, x_1, \ldots, x_N]$.
[The word *form* is interchangeable with *homogeneous polynomial*.]

Example

In $\mathbb{C}[X, Y, Z]$ such an ideal is $I = (XZ, YZ, X^3 - 3X^2Y - XY^2)$.
A non-example is $J = (X^2 - Y, Z^2)$.

Ordinary Powers

Given a ideals $I, J \subseteq R$, we may multiply ideals. Recall:

$$IJ = (FG \ : \ F \in I, \ G \in J).$$

We may extend this to (ordinary) powers:

$$I^r = (G_{i_1} G_{i_2} \cdots G_{i_r} \ : \ G_i \in I)$$

Example

Let $I = (X, Y) \subseteq \mathbb{C}[X, Y, Z]$. Then $I^2 = (X^2, XY, Y^2)$, $I^3 = (X^3, X^2Y, XY^2, Y^3)$, etc.

Note: We have $I^r \subseteq I^t$ if and only if $r \geq t$. *ideal gets (strictly) smaller*

Symbolic Powers

Definition

Given an ideal $I \subseteq R$, we define the m-th symbolic power of I to be

$$I^{(m)} = R \cap \left(\bigcap_{P} (I^m R_P) \right).$$

This can reduce to a much cleaner definition if more information about I is available.

Note: We have $I^{(r)} \subseteq I^{(t)}$ if and only if $r \geq t$.

Ordinary vs. Symbolic

Question

What is the relationship between I^r and $I^{(m)}$?

Answer: It depends on I.

A partial answer: $I^r \subseteq I^{(m)}$ if and only if $r \geq m$.

A (further) partial answer: $I^{(m)} \subseteq I^r$ implies $m \geq r$.

Before elaborating, we ask: what can symbolic powers look like?

Symbolic Powers of Edge Ideals

First studied by R. Villareal in the 1990s

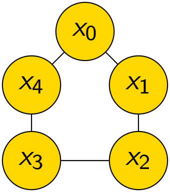
Let $V = \{x_1, x_2, \ldots, x_n\}$ be a set of variables and consider the (simple) graph $G = (V, E)$, where E contains 2-element sets comprised of pairs of the variables (so, e.g., $\{x_1, x_2\} \in E$ but $\{x_1, x_2, x_3\}, \{x_1^2\} \notin E$).

Definition

Given $G = (V, E)$ as above, the edge ideal of G is $I(G) = (x_i x_j : \{x_i, x_j\} \in E) \subseteq k[x_1, x_2, \ldots, x_n]$.

Fact: For an edge ideal I, $I^{(m)} = \bigcap_i P_i^m$, where the P_i correspond to minimal vertex covers of G.

$I = I(C_5) = (x_0x_1, x_1x_2, x_2x_3, x_3x_4, x_4x_0)$



Here, the ring is $R = k[x_0, x_1, x_2, x_3, x_4]$, and the ideals corresponding to minimal vertex covers are $P_1 = (x_0, x_1, x_3)$, $P_2 = (x_0, x_2, x_3)$, $P_3 = (x_0, x_2, x_4)$, $P_4 = (x_1, x_2, x_4)$, $P_5 = (x_1, x_3, x_4)$. Then

$$\begin{aligned} I^{(2)} &= P_1^2 \cap P_2^2 \cap P_3^2 \cap P_4^2 \cap P_5^2 \\ &= (x_0^2x_1^2, x_0x_1^2x_2, x_1^2x_2^2, x_0x_1x_2x_3, x_1x_2^2x_3, x_2^2x_3^2, x_0^2x_1x_4, x_0x_1x_2x_4, \\ &\qquad x_0x_1x_3x_4, x_0x_2x_3x_4, x_1x_2x_3x_4, x_2x_3^2x_4, x_0^2x_4^2, x_0x_3x_4^2, x_3^2x_4^2) \\ &= I^2. \end{aligned}$$

But $I^{(t)} \neq I^t$ for all $t > 2$.

Bipartite edge ideal characterization

Theorem (Simis-Vasconcelos-Villareal (1994))

Given an edge ideal $I = I(G) \subseteq k[x_1, x_2, \ldots, x_n]$ as above, the following are equivalent.

(i) $I^{(m)} = I^m$ *for all* $m \geq 1$.

(ii) *The graph G is bipartite.*

Edge ideals of non-bipartite graphs

A consequence of the previous theorem is: if G is not bipartite and $I = I(G)$, then there exists a $t > 0$ such that $I^{(t)} \neq I^t$.

Our main question:

Problem

If $I = I(G)$ and G is not bipartite, how do $I^{(m)}$ and I^r compare?

Problem (Invariant Problem)

Compute invariants related to the containment $I^{(m)} \subseteq I^r$.

A conjecture

Focus of the honors project at Dordt College in Spring 2015: what happens when G is not bipartite?

Conjecture (Ellis–Wilson–McLoud-Mann)

Let $I = I(C_{2n+1}) \subseteq k[x_1, \ldots, x_{2n+1}]$ be the edge ideal of the odd cycle on $2n+1$ vertices. Then

- $I^t = I^{(t)}$ *for all* $1 \leq t \leq n$*;*
- $I^t \neq I^{(t)}$ *for all* $t > n$*.*

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Of importance when discussing ideal containments is the *initial degree*.

Definition

Let $J \subsetneq k[x_0, x_1, \ldots, x_N]$ *be a nonzero homogeneous ideal. Define*

$$\alpha(J) = \min\{d \,:\, \textit{there exists } 0 \neq f \in J, \deg(f) = d\}.$$

Note: if $\alpha(I^{(m)}) < \alpha(I^r)$ then $I^{(m)} \not\subseteq I^r$.

Example

Given an edge ideal $I = I(G)$, $\alpha(I) = 2$ and $\alpha(I^r) = r\alpha(I) = 2r$. Computing $\alpha(I^{(m)})$ is more delicate.

Given I, the edge ideal of C_{2n+1},

$$\alpha(I^{(m)}) = 2m - \lfloor \frac{m}{n+1} \rfloor$$

Half-proof of the conjecture

Proposition

Let $I = I(C_{2n+1}) \subseteq k[x_1, \ldots, x_{2n+1}]$ be the edge ideal of the odd cycle on $2n+1$ vertices. Then $I^{(t)} \neq I^t$ for all $t > n$.

Proof.

We know $\alpha(I^t) = 2t$ and $\alpha(I^{(t)}) = 2t - \lfloor \frac{t}{n+1} \rfloor \leq 2t - 1 < 2t$ when $t > n$. □

Our work attempting to prove the rest of the conjecture is ongoing.

Thanks

Thank you!