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Much More Than Symbolics: The Early History of Algebra and Its Significance for Introductory Algebra Education

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Disciplines

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Comments

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OUTLINE

✠ Textbook Project Description in Brief

- Course for Prospective Middle School Mathematics Teachers
  - Curricular Identity Crisis: Changing Educational Scene
  - Refocused Content, Drawing from History of Mathematics
    - Focus on origins and historical development of key ideas
    - Global organization: mathematical topics
    - Internal sequencing: culture and chronology
- Need for Class Materials
  - Extensive Outlines
  - Homemade Exercises
- Textbook Production: Sections in Process
  - Audience: Middle School Mathematics Teachers
  - Focus on Educationally Significant Developments
  - Historical Resources: Synthesize Current Research

- Educational Benefits for (Prospective) Mathematics Teachers
  - Cultural
    - Acknowledge broader cultural matrix of mathematics
    - Appreciate multi-cultural contributions to mathematics
    - Contribute to integrated social studies units
  - Philosophical
    - Acquire more realistic view of the nature of mathematics
  - Curricular & Pedagogical: Developmental Perspective
    - Ontogeny recapitulates phylogeny?
    - Cognitive development, but with significant cultural differences
    - Recognize developmental prerequisites
    - Gain deepened understanding of meaning
    - Identify potential cognitive obstacles
  - Enrichment Activities
    - Draw on alternative understandings and procedures
    - Incorporate supplementary activities
    - Explore simple primary source material

Table of Contents

✠ What is (Elementary) Algebra? Some Tacit Definitions

- Popular & Technical Definitions

  Algebra is a field that . . .

  - manipulates letters like numbers.
  - solves problems by doing symbolic calculations on equations.
  - generalizes arithmetic.
  - creates and explores abstract quantitative structures.

- Definitions Put Forward in Educational Research

  Algebra is a field that . . .

  - studies symbolic representation of problems using letters.
  - studies procedures for simplifying and solving equations.
  - studies general relationships among numbers.
  - studies theories of quantitative structures.

- Definitions Adopted/Assumed by Histories of Mathematics
  - Conventional periodization: Rhetorical, Syncopated, Symbolic
    - Assumes symbolism is central
    - Values developments for advancing symbolic representation
    - Treats early algebra as proto-algebra at best
    - Ironically, denies algebra-level status for Islamic contributions
- Alternative Definitions Informed by History of Mathematics
  - Open to features that transcend and contextualize notation
  - Propose a semi-stable core with changing manifestations
    - Algebra is the premier problem-solving instrument.
    - Arithmetic calculates outputs; algebra solves for inputs.
    - Algebra treats unknowns in the same way as known quantities.
    - Algebra develops efficient systematic methods for problem solving.
- Proper Focus for Elementary Algebra Education
  - Focus less on symbolics at the outset
  - Pay more attention to concrete problem-solving strategies
  - Introduce symbolism as a powerful problem-solving strategy

✠ Simple and Concrete Problem Solving

- Some Sample Problems: Is This Arithmetic or Algebra?
  - Problem 1
    A work supervisor has a crew of eight. He pays each person with three loaves of bread. How many loaves does he distribute?
    An arithmetic problem: $8 \times 3 = 24$.
  - Problem 2
    A work supervisor receives 20 loaves of bread for his crew. He stores them on three equal-sized racks but has to leave two off. How many are on each rack?
    An algebra problem? $20 - 2 = 18; 18 \div 3 = 6$.
  - Problem 3
    A surveyor measures a rectangular field, recording the area as 60 and the diagonals' length as 13. What are the dimensions of the field?
    An algebra problem: Pythagorean triple $(5, 12, 13)$; or compute: $13^2 + 2 \times 60 = 289$; $13^2 - 2 \times 60 = 49$; $(\sqrt{289} \pm \sqrt{49})/2 = 5, 12$.

- Arithmetic Problem Solving for Unknown Input
  - *Trial-and-Error Guessing*
    - Guess solution; adjust it up or down, depending on match-up
  - *Method of False Position*: Systematic Guessing
    - Guess solution; scale it by (true output) : (calculated output)
    - Exhibits understanding of problem's proportional relationships
    - Used by various ancient cultures
    - Chinese *Method of Excess and Deficit*: Double False Position
  - Arithmetic-in-Reverse Solutions
    - Solving problems by simple inverse calculations (e.g., division)
    - Indian *Method of Inversion*:
      reverse the calculation sequence, using inverse operations
      matches what we do with symbolic algebra
    - Example: three times a quantity plus four yields ten.
      $[Q \to 3Q \to 3Q + 4 = 10]$; $10 - 4 = 6$; $6 \div 3 = 2$
      $3x + 4 = 10$; $3x = 10 - 4 = 6$; $x = 6 \div 3 = 2$

- Concrete and Geometric Modeling
  - Egyptian *Aha* Problem: RMP #26
    A quantity plus its fourth-part gives 15. Find the quantity.
  - Egyptian Solution: Algorithmic Calculation Sequence
    Calculate with 4: $4 + 1 = 5$.
    Divide: $15 \div 5 = 3$.
    Multiply: $\underline{4} \times 3 = 12$, the quantity.
  - Egyptian Solution: *Method of Part-Size*
    Make a fourth-part the unit: $4 + 1$ parts. 
    5 parts are 15; 1 part is 3.  $= 15$;  $= 3$
    The full quantity is 4 times as much: 12.  $= 12$
  - *Method of False Position*
    Guess 4: $4 + 1 = 5$; $15 \div 5 = 3$; $\underline{3} \times 4 = 12$
    Not exactly the Egyptian scribe's solution: bad guess; wrong final multiplication; counterfactual vs. concrete reasoning.

- Is the Egyptian *Method of Part-Size* Algebraic? Yes.
  - It's a systematic method of problem solving.
  - It solves for an unknown input.
  - It treats the unknown input as a quantity one can calculate with.
- Algebraic features of the Egyptian *Method of Part-Size*
  - It essentially introduces a new unknown (a fourth-part) and so reconfigures the problem to be solved.
  - It concretely matches a symbolic solution method we might use.

$$x + \frac{x}{4} = 15; \qquad 4\left(\frac{x}{4}\right) + \left(\frac{x}{4}\right) = 15; \qquad 5\left(\frac{x}{4}\right) = 15$$

$$\left(\frac{x}{4}\right) = 15 \div 5 = 3; \qquad x = 4\left(\frac{x}{4}\right) = 4 \times 3 = 12$$

- Old Babylonian Linear Problem: TMS VII
  The length of a rectangle plus one-fourth its width is 7. The length plus the width is 10. Find the length and width.

- Babylonian Solution: Algorithmic Calculation Sequence
  (1) Multiply: $4 \times 7 = 28$.
  (2) Subtract: $28 - 10 = 18$.
  (3) Divide: $18 \div 3 = 6$, the length.
  (4) Subtract: $10 - 6 = 4$, the width.

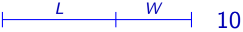
- Concrete Interpretation; Associated Symbolic Algebra

(0)	Length plus one-fourth width is 7;	$L + \frac{1}{4}W = 7$
	Length plus width is 10.	$L + W = 10$
(1)	Scale: four lengths plus one width is 28.	$4L + W = 28$
(2)	Subtract: three lengths is 18.	$3L = 18$
(3)	Divide: one length is 6.	$L = 6$
(4)	Subtract: one width is 4.	$W = 4$

- Babylonian Solution's Underlying Geometric Reasoning
  - (0) Length plus one-fourth width is 7;

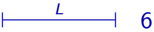
  - (1) Scale: four lengths plus one width is 28.

  - (0) Length plus width is 10.



  - (2) Subtract: three lengths is 18.

  - (3) Divide: one length is 6.



  - (4) Subtract: one width is 4.

- Is the Babylonian Solution Method Algebraic? Yes.
  - It's a systematic method of problem solving.
  - It solves for unknown inputs.
  - It treats unknown inputs as quantities one can calculate with.
- Algebraic features of the Babylonian Solution Method
  - It doesn't solve for one unknown and substitute to reduce the data to one complex equation in one unknown.
  - It arithmetically combines equations and eliminates unknowns.
  - It concretely matches a symbolic linear algebra solution method.

$$L + \frac{W}{4} = 7 \qquad 4L + W = 28$$

$$L + W = 10 \qquad L + W = 10 \qquad L + W = 10$$

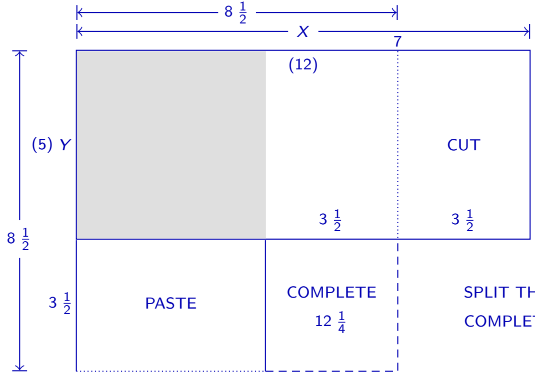
$$3L = 18 \qquad L = 6$$

$$W = 4$$

✠ Geometric Quadratic Algebra

- Babylonian Quadratic Algebra
  - Babylonian Quadratic Problem: YBC 6967
    A reciprocal exceeds its reciprocal by 7. Find the two numbers.
    [Note: the product of a reciprocal pair is taken as 60.]
  - Babylonian Solution: Algorithmic Calculation Sequence
    (1) Halve: $7 \div 2 = 3\,\frac{1}{2}$.
    (2) Square: $3\,\frac{1}{2} \times 3\,\frac{1}{2} = 12\,\frac{1}{4}$.
    (3) Add: $12\,\frac{1}{4} + 60 = 72\,\frac{1}{4}$.
    (4) Take the square root: $\sqrt{72\frac{1}{4}} = 8\,\frac{1}{2}$.
    (5) Add and subtract: $8\,\frac{1}{2} + 3\,\frac{1}{2} = 12$, $8\,\frac{1}{2} - 3\,\frac{1}{2} = 5$.
    These are the numbers sought: $12 \times 5 = 60$, $12 - 5 = 7$.

- Babylonian Solution's Underlying Geometry



$XY = 60$

SPLIT THE DIFFERENCE

COMPLETE THE SQUARE

$$X - Y = 7 \qquad 7 \div 2 = 3\,\tfrac{1}{2}$$

$$3\,\tfrac{1}{2} \times 3\,\tfrac{1}{2} = 12\,\tfrac{1}{4} \qquad 12\,\tfrac{1}{4} + 60 = 72\,\tfrac{1}{4}$$

$$\sqrt{72\tfrac{1}{4}} = 8\,\tfrac{1}{2} \qquad 8\,\tfrac{1}{2} \pm 3\,\tfrac{1}{2} = 12, 5$$

- Symbolic Formulation of Algorithm

(0) $XY = P; \quad X - Y = D.$
Combining by substituting, $X(X - D) = X^2 - DX = P.$

(1) $D \div 2 = \dfrac{D}{2}.$

(2) $\left(\dfrac{D}{2}\right)^2 = \dfrac{D^2}{4}.$

(3) $\dfrac{D^2}{4} + P$

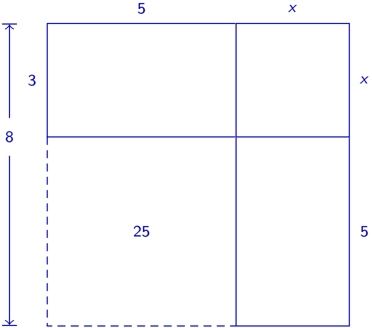
(4) $\sqrt{\dfrac{D^2}{4} + P}$

(5) $X, Y = \sqrt{\dfrac{D^2}{4} + P} \pm \dfrac{D}{2}$: a version of the Quadratic Formula.

- Are Babylonian Quadratic Solution Methods Algebraic? Yes.
  - They give systematic methods of problem solving.
  - They solve for unknown inputs.
  - They treat unknown inputs as quantities one can calculate with.
- Algebraic features of Babylonian Quadratic Solution Methods
  - They solve problems using effective geometric cut-and-paste complete-the-square methods.
  - They concretely match symbolic algebraic solution methods.
  - Their methods do *not* give a version of the quadratic formula; instead, they give the equivalent of our complete-the-square procedure.
  - The Babylonian *Complete-The-Square Method* is historically very important: this is what converted algebra from mere recreational problem solving into the art and science of problem solving.

- Islamic Quadratic Algebra
  - Islamic Quadratic Problem: al-Khwārizmī, Type 4 Equation
    Squares and roots equal to numbers: one square and ten of its roots equals thirty-nine.
  - al-Khwārizmī's Solution: Algorithmic Procedure (all in words)
    (1) Halve the number of roots: five.
    (2) Multiply this by itself: twenty-five.
    (3) Add this to thirty-nine: sixty-four.
    (4) Take this number's root: eight.
    (5) Subtract half the original roots: three.
    (6) Three is the root of the square sought;
    the square itself is nine.

- Geometric Demonstration of Solution



Draw a square,
and add rectangles of length 5
to two sides of the square.
Total area, as given, is 39.
Complete the square;
25 is added to the area, giving 64.
The side of the large square is 8;
the small square's side is 5 less: 3.

$$x^2 + 10x = 39$$
$$x^2 + 10x + 25 = 39 + 25 = 64$$
$$(x+5)^2 = 8^2$$
$$x + 5 = 8$$
$$x = 8 - 5 = 3$$

- Symbolic Formulation of Algorithm

(0) $x^2 + bx = c$

(1) $b \div 2 = \frac{b}{2}$

(2) $\left(\frac{b}{2}\right)^2 = \frac{b^2}{4}$

(3) $\frac{b^2}{4} + c$

(4) $\sqrt{\frac{b^2}{4} + c}$

(5) $x = \sqrt{\frac{b^2}{4} + c} - \frac{b}{2}$: a version of the Quadratic Formula.

- Are Islamic Quadratic Solution Methods Algebraic? Yes.
  - They systematically solve a wide variety of elementary problems.
  - They solve for unknown inputs.
  - They treat unknown inputs as quantities one can calculate with.
  - They perform calculations with algebraic expressions and manipulate equations, albeit verbally.
- Algebraic features of Islamic Quadratic Solution Methods
  - Elementary algebra is systematically organized into a discipline: equations are fully enumerated and categorized, effective solution procedures are developed and their legitimacy geometrically demonstrated, algebraic calculations are explored, and applications are made.
  - Their solution algorithms match symbolic algebraic solution procedures.
  - However, solution methods are not yet fully uniform, nor do they generate formulas.
  - Islamic algebra becomes the springboard for further developments in later European circles.

✠ Summary: Educational Lessons from HoM

- Norm of Concrete/Holistic Beginnings
  - Begin concretely, keeping the problem-solving goal of algebra centrally in mind
  - Model problems appropriately, using a variety of concrete approaches and effective procedures
- Norm of Progressive Development
  - Use more complex models and procedures as needed and as students are ready for them
  - Introduce symbolic abstraction and operations gradually, in parallel with concrete representation and manipulations
- Norm of Seeking Efficient Uniform Procedures (Later Talk)
  - Reveal the limitations of a narrowly concrete approach
  - Demonstrate the power and simplicity of an abstract symbolic approach

- Value of History of Mathematics for Learning Algebra
  - Suggests ways to highlight/explain key ideas, avoid difficulties
  - Offers both curricular and pedagogical insights
  - Provides enrichment and exploratory materials

- Questions or Suggestions?